

Patterning and Algebra, Grades 4 to 6

**A Guide to Effective Instruction
in Mathematics,
Kindergarten to Grade 6**

2008

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Every effort has been made in this publication to identify mathematics resources and tools (e.g., manipulatives) in generic terms. In cases where a particular product is used by teachers in schools across Ontario, that product is identified by its trade name, in the interests of clarity. Reference to particular products in no way implies an endorsement of those products by the Ministry of Education.

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INTRODUCTION

Patterning and Algebra, Grades 4 to 6 is a practical guide that teachers will find useful in helping students to achieve the curriculum expectations outlined for Grades 4 to 6 in the Patterning and Algebra strand of *The Ontario Curriculum, Grades 1–8: Mathematics, 2005*. This guide provides teachers with practical applications of the principles and theories that are elaborated in *A Guide to Effective Instruction in Mathematics, Kindergarten to Grade 6, 2006*.

The first part of the guide provides a detailed discussion of the two “big ideas”, or major mathematical themes, in Patterning and Algebra, and provides a discussion of mathematical models and instructional strategies that have proved effective in helping students understand the mathematical concepts related to each big idea. The guide emphasizes the importance of focusing on the big ideas in mathematical instruction to achieve the goal of *helping students gain a deeper understanding of mathematical concepts*. At the end of the first part of the guide is a list of references cited.

The second part of the guide provides sample learning activities, for Grades 4, 5, and 6, that illustrate how a learning activity can be designed to:

- focus on an important curriculum topic;
- involve students in applying the seven mathematical processes described in the mathematics curriculum document;
- develop understanding of the big ideas in Patterning and Algebra.

At the end of the second part of the guide is an appendix that discusses assessment strategies for teachers. There is also a glossary that includes mathematical and other terms that are used in the guide.

The Pleasure of Mathematical Surprise and Insight

Young children enter school mathematically curious, imaginative, and capable. They have to learn to be otherwise (Papert, 1980). The aim of this resource is to help consolidate and extend junior students’ mathematical capacity and their potential for mathematical growth by providing ideas and classroom activities that draw their attention to relationships embedded in the “big ideas” of Patterning and Algebra and that offer them opportunities to experience the pleasure of mathematical surprise and insight (Gadanidis, 2004).

The activities in this resource incorporate the ideas and practice of classroom teachers. The activities have been field-tested in Ontario classrooms, and feedback from practicing teachers has been used to create the final versions. The chapter “The “Big Ideas” of Patterning and Algebra” (pp. 15–21) discusses the big ideas on which the activities have been built and contains additional ideas for classroom activities.

The teaching of mathematics around big ideas offers students opportunities to develop a sophisticated understanding of mathematics concepts and processes, and helps them to maintain their interest in and excitement about doing and learning mathematics.

Working Toward Equitable Outcomes for Diverse Students

All students, whatever their socio-economic, ethnocultural, or linguistic background, must have opportunities to learn and to grow, both cognitively and socially. When students can make personal connections to their learning, and when they feel secure in their learning environment, their true capacity will be realized in their achievement. A commitment to equity and inclusive instruction in Ontario classrooms is therefore critical to enabling all students to succeed in school and, consequently, to become productive and contributing members of society.

To create effective conditions for learning, teachers must take care to avoid all forms of bias and stereotyping in resources and learning activities, which can quickly alienate students and limit their learning. Teachers should be aware of the need to provide a variety of experiences and to encourage multiple perspectives, so that the diversity of the class is recognized and all students feel respected and valued. Learning activities and resources for teaching mathematics should be inclusive, providing examples and illustrations and using approaches that recognize the range of experiences of students with diverse backgrounds, knowledge, skills, interests, and learning styles.

The following are some strategies for creating a learning environment that acknowledges and values the diversity of students and enables them to participate fully in the learning experience:

- providing mathematics problems with situations and contexts that are meaningful to all students (e.g., problems that reflect students’ interests, home-life experiences, and cultural backgrounds and that arouse their curiosity and spirit of enquiry);
- using mathematics examples drawn from diverse cultures, including Aboriginal peoples;
- using children’s literature that reflects various cultures and customs as a source of mathematical examples and situations;
- understanding and acknowledging customs and adjusting teaching strategies as necessary. For example, a student may come from a culture in which it is considered inappropriate for a child to ask for help, express opinions openly, or make direct eye contact with an adult;
- considering the appropriateness of references to holidays, celebrations, and traditions;
- providing clarification if the context of a learning activity is unfamiliar to students (e.g., describing or showing a food item that may be new to some students);

- evaluating the content of mathematics textbooks, children’s literature, and supplementary materials for cultural or gender bias;
- designing learning and assessment activities that allow students with various learning styles (e.g., auditory, visual, tactile/kinaesthetic) to participate meaningfully;
- providing opportunities for students to work both independently and interdependently with others;
- providing opportunities for students to communicate orally and in writing in their home language (e.g., pairing English language learners with a first-language peer who also speaks English);
- using diagrams, pictures, manipulatives, sounds, and gestures to clarify mathematical vocabulary that may be new to English language learners.

For a full discussion of equity and diversity in the classroom, as well as a detailed checklist for providing inclusive mathematics instruction, see pages 34–40 in Volume 1 of *A Guide to Effective Instruction in Mathematics, Kindergarten to Grade 6, 2006*.

Accommodations and Modifications

The learning activities in this document have been designed for students with a range of learning needs. Instructional and assessment tasks are open-ended, allowing most students to participate fully in learning experiences. In some cases, individual students may require *accommodations* and/or *modifications*, in accordance with their Individual Education Plan (IEP), to support their participation in learning activities.

PROVIDING ACCOMMODATIONS

Students may require accommodations, including special strategies, support, and/or equipment, to allow them to participate in learning activities. There are three types of accommodations:

- *Instructional accommodations* are adjustments in teaching strategies, including styles of presentation, methods of organization, or the use of technology or multimedia.
- *Environmental accommodations* are supports or changes that the student may require in the physical environment of the classroom and/or the school, such as preferential seating or special lighting.
- *Assessment accommodations* are adjustments in assessment activities and methods that enable the student to demonstrate learning, such as allowing additional time to complete tasks or permitting oral responses to test questions.

The term accommodations is used to refer to the special teaching and assessment strategies, human supports, and/or individualized equipment required to enable a student to learn and to demonstrate learning. Accommodations do not alter the provincial curriculum expectations for the grade.

Modifications are changes made in the age-appropriate grade-level expectations for a subject . . . in order to meet a student’s learning needs. These changes may involve developing expectations that reflect knowledge and skills required in the curriculum for a different grade level and/or increasing or decreasing the number and/or complexity of the regular grade-level curriculum expectations.

(Ontario Ministry of Education, 2004, pp. 25–26)

Some of the ways in which teachers can provide accommodations with respect to mathematics learning activities are listed in the following chart.

Instructional Accommodations

- Vary instructional strategies, using different manipulatives, examples, and visuals (e.g., concrete materials, pictures, diagrams) as necessary to aid understanding.
- Rephrase information and instructions to make them simpler and clearer.
- Use non-verbal signals and gesture cues to convey information.
- Teach mathematical vocabulary explicitly.
- Have students work with a peer.
- Structure activities by breaking them into smaller steps.
- Model concepts using concrete materials and computer software, and encourage students to use them when learning concepts or working on problems.
- Have students use calculators and/or addition and multiplication grids for computations.
- Format worksheets so that they are easy to understand (e.g., use large-size font; an uncluttered layout; spatial cues, such as arrows; colour cues).
- Encourage students to use graphic organizers and graph paper to organize ideas and written work.
- Provide augmentative and alternative communications systems.
- Provide assistive technology, such as text-to-speech software.
- Provide time-management aids (e.g., checklists).
- Encourage students to verbalize as they work on mathematics problems.
- Provide access to computers.
- Reduce the number of tasks to be completed.
- Provide extra time to complete tasks.

Environmental Accommodations

- Provide an alternative work space.
- Seat students strategically (e.g., near the front of the room; close to the teacher in group settings; with a classmate who can help them).
- Reduce visual distractions.
- Minimize background noise.
- Provide a quiet setting.
- Provide headphones to reduce audio distractions.
- Provide special lighting.
- Provide assistive devices or adaptive equipment.

Assessment Accommodations

- Have students demonstrate understanding using concrete materials, computer software, or orally rather than in written form.
- Have students record oral responses on audiotape.
- Have students' responses on written tasks recorded by a scribe.
- Provide assistive technology, such as speech-to-text software.
- Provide an alternative setting.
- Provide assistive devices or adaptive equipment.
- Provide augmentative and alternative communications systems.
- Format tests so that they are easy to understand (e.g., use large-size font; an uncluttered layout; spatial cues, such as arrows; colour cues).
- Provide access to computers.
- Provide access to calculators and/or addition and multiplication grids.
- Provide visual cues (e.g., posters).
- Provide extra time to complete problems or tasks or answer questions.
- Reduce the number of tasks used to assess a concept or skill.

MODIFYING CURRICULUM EXPECTATIONS

Students who have an IEP may require modified expectations, which differ from the regular grade-level curriculum expectations. When developing modified expectations, teachers make important decisions regarding the concepts and skills that students need to learn.

Most of the learning activities in this document can be adapted for students who require modified expectations. The following chart provides examples of how a teacher could deliver learning activities that incorporate individual students' modified expectations.

Modified Program	What It Means	Example
<i>Modified learning expectations, same activity, same materials</i>	The student with modified expectations works on the same or a similar activity, using the same materials.	The learning activity involves describing the rule for a growth pattern of linking cubes. Students with modified expectations demonstrate their understanding of the rule by extending the pattern.
<i>Modified learning expectations, same activity, different materials</i>	The student with modified expectations engages in the same activity, but uses different materials that enable him/her to remain an equal participant in the activity.	The learning activity involves creating growth patterns using numbers or diagrams. Students with modified expectations may also use linking cubes or colour tiles to represent patterns.

(continued)

Modified Program	What It Means	Example
<i>Modified learning expectations, different activity, different materials</i>	Students with modified expectations participate in different activities.	Students with modified expectations work on patterning and algebra activities that reflect their learning expectations, using a variety of concrete materials.

(Adapted from *Education for All: The Report of the Expert Panel on Literacy and Numeracy Instruction for Students With Special Education Needs, Kindergarten to Grade 6*, p. 119.)

It is important to note that some students may require both accommodations and modified expectations.

The Mathematical Processes

The Ontario Curriculum, Grades 1–8: Mathematics, 2005 identifies seven mathematical processes through which students acquire and apply mathematical knowledge and skills. The mathematical processes that support effective learning in mathematics are as follows:

- problem solving
- connecting
- reasoning and proving
- representing
- reflecting
- communicating
- selecting tools and computational strategies

The learning activities in this guide demonstrate how the mathematical processes help students develop mathematical understanding. Opportunities to solve problems, to reason mathematically, to reflect on new ideas, and so on, make mathematics meaningful for students. The learning activities also demonstrate that the mathematical processes are interconnected – for example, problem-solving tasks encourage students to represent mathematical ideas, to select appropriate tools and strategies, to communicate and reflect on strategies and solutions, and to make connections between mathematical concepts.

Problem Solving: Each of the learning activities is structured around a problem or an inquiry. As students solve problems or conduct investigations, they make connections between new mathematical concepts and ideas that they already understand. The focus on problem solving and inquiry in the learning activities also provides opportunities for students to:

- find enjoyment in mathematics;
- develop confidence in learning and using mathematics;
- work collaboratively and talk about mathematics;
- communicate ideas and strategies;

- reason and use critical thinking skills;
- develop processes for solving problems;
- develop a repertoire of problem-solving strategies;
- connect mathematical knowledge and skills with situations outside the classroom.

Reasoning and Proving: The learning activities described in this document provide opportunities for students to reason mathematically as they explore new concepts, develop ideas, make mathematical conjectures, and justify results. The learning activities include questions that teachers can use to encourage students to explain and justify their mathematical thinking, and to consider and evaluate the ideas proposed by others.

Reflecting: Throughout the learning activities, students are asked to think about, reflect on, and monitor their own thought processes. For example, questions posed by the teacher encourage students to think about the strategies they use to solve problems and to examine mathematical ideas that they are learning. In the Reflecting and Connecting part of each learning activity, students have an opportunity to discuss, reflect on, and evaluate their problem-solving strategies, solutions, and mathematical insights.

Selecting Tools and Computational Strategies: Mathematical tools, such as manipulatives, pictorial models, and computational strategies, allow students to represent and do mathematics. The learning activities in this document provide opportunities for students to select tools (concrete, pictorial, and symbolic) that are personally meaningful, thereby allowing individual students to solve problems and to represent and communicate mathematical ideas at their own level of understanding.

Connecting: The learning activities are designed to allow students of all ability levels to connect new mathematical ideas to what they already understand. The learning activity descriptions provide guidance to teachers on ways to help students make connections between concrete, pictorial, and symbolic mathematical representations. Advice on helping students develop conceptual understanding is also provided. The problem-solving experience in many of the learning activities allows students to connect mathematics to real-life situations and meaningful contexts.

Representing: The learning activities provide opportunities for students to represent mathematical ideas by using concrete materials, pictures, diagrams, numbers, words, and symbols. Representing ideas in a variety of ways helps students to model and interpret problem situations, understand mathematical concepts, clarify and communicate their thinking, and make connections between related mathematical ideas. Students' own concrete and pictorial representations of mathematical ideas provide teachers with valuable assessment information about student understanding that cannot be assessed effectively using paper-and-pencil tests.

Communicating: Communication of mathematical ideas is an essential process in learning mathematics. Throughout the learning activities, students have opportunities to express mathematical ideas and understandings orally, visually, and in writing. Often, students are asked to work in pairs or in small groups, thereby providing learning situations in which students talk about the mathematics that they are doing, share mathematical ideas, and ask clarifying questions of their classmates. These oral experiences help students to organize their thinking before they are asked to communicate their ideas in written form.

Addressing the Needs of Junior Learners

Every day, teachers make many decisions about instruction in their classrooms. To make informed decisions about teaching mathematics, teachers need to have an understanding of the big ideas in mathematics, the mathematical concepts and skills outlined in the curriculum document, effective instructional approaches, and the characteristics and needs of learners.

The following chart outlines general characteristics of junior learners, and describes some of the implications of these characteristics for teaching mathematics to students in Grades 4, 5, and 6.

Area of Development	Characteristics of Junior Learners	Implications for Teaching Mathematics
Intellectual Development	<p>Generally, students in the junior grades:</p> <ul style="list-style-type: none"> • prefer active learning experiences that allow them to interact with their peers; • are curious about the world around them; • are at a concrete, operational stage of development, and are often not ready to think abstractly; • enjoy and understand the subtleties of humour. 	<p>The mathematics program should provide:</p> <ul style="list-style-type: none"> • learning experiences that allow students to actively explore and construct mathematical ideas; • learning situations that involve the use of concrete materials; • opportunities for students to see that mathematics is practical and important in their daily lives; • enjoyable activities that stimulate curiosity and interest; • tasks that challenge students to reason and think deeply about mathematical ideas.
Physical Development	<p>Generally, students in the junior grades:</p> <ul style="list-style-type: none"> • experience a growth spurt before puberty (usually at age 9–10 for girls, at age 10–11 for boys); • are concerned about body image; • are active and energetic; • display wide variations in physical development and maturity. 	<p>The mathematics program should provide:</p> <ul style="list-style-type: none"> • opportunities for physical movement and hands-on learning; • a classroom that is safe and physically appealing.

Area of Development	Characteristics of Junior Learners	Implications for Teaching Mathematics
Psychological Development	<p>Generally, students in the junior grades:</p> <ul style="list-style-type: none"> • are less reliant on praise, but still respond well to positive feedback; • accept greater responsibility for their actions and work; • are influenced by their peer groups. 	<p>The mathematics program should provide:</p> <ul style="list-style-type: none"> • ongoing feedback on students' learning and progress; • an environment in which students can take risks without fear of ridicule; • opportunities for students to accept responsibility for their work; • a classroom climate that supports diversity and encourages all members to work cooperatively.
Social Development	<p>Generally, students in the junior grades:</p> <ul style="list-style-type: none"> • are less egocentric, yet require individual attention; • can be volatile and changeable in regard to friendship, yet want to be part of a social group; • can be talkative; • are more tentative and unsure of themselves; • mature socially at different rates. 	<p>The mathematics program should provide:</p> <ul style="list-style-type: none"> • opportunities to work with others in a variety of groupings (pairs, small groups, large group); • opportunities to discuss mathematical ideas; • clear expectations of what is acceptable social behaviour; • learning activities that involve all students regardless of ability.
Moral and Ethical Development	<p>Generally, students in the junior grades:</p> <ul style="list-style-type: none"> • develop a strong sense of justice and fairness; • experiment with challenging the norm and ask "why" questions; • begin to consider others' points of view. 	<p>The mathematics program should provide:</p> <ul style="list-style-type: none"> • learning experiences that provide equitable opportunities for participation by all students; • an environment in which all ideas are valued; • opportunities for students to share their own ideas, and evaluate the ideas of others.

Adapted, with permission, from *Making Math Happen in the Junior Years* (Elementary Teachers' Federation of Ontario, 2004).

Learning About Patterning and Algebra in the Junior Grades

The development of an understanding of patterning and algebra concepts and relationships is a gradual one, moving from experiential and physical learning to theoretical and inferential learning. Patterning and algebra thinking in the junior years begins to bridge the two.

PRIOR LEARNING

In the primary grades, students learn to identify patterns in shapes, designs, and movement, as well as in sets of numbers. They study both repeating patterns and growing and shrinking patterns and develop ways to extend them. Students use concrete materials and pictorial displays to create patterns and recognize relationships. Through the observation of different representations of a pattern, students begin to identify some of the properties of the pattern. Students develop an understanding of the concept of equality between pairs of expressions, using concrete materials and addition and subtraction of one- and two-digit numbers. Students also learn to solve simple missing-number equations.

Experiences in the primary classroom include identifying and describing patterns through investigation in real-life settings, with concrete materials and diagrams, and with numbers. Experiences also include students communicating about patterning and algebra using words, diagrams, symbols, and concrete materials.

KNOWLEDGE AND SKILLS DEVELOPED IN THE JUNIOR GRADES

In the junior grades, students extend their understanding of patterning and algebra by: generating patterns that involve addition, subtraction, multiplication, or division using one- or two-digit numbers, or that involve reflections, translations, or rotations; determining the missing numbers in equations involving multiplication of one- and two-digit numbers using concrete materials and guess and check; and using the commutative and distributive properties to facilitate computation. Students investigate variables as unknown quantities and demonstrate equality using multiplication or division in equations with unknown quantities on both sides. They also represent patterns using ordered pairs and graphs and learn to calculate any term in a pattern when given the term number.

As is the case in the primary grades, experiences in the junior classroom involve investigation in real-life settings, with concrete materials and diagrams, and with numbers, and they include students communicating using words, diagrams, symbols, and concrete materials. Investigations such as the following help students to start to develop a more abstract understanding of patterning and algebra concepts. **Sample problem:** Use the method of your choice to determine the value of the variable in the equation $2 \times n + 3 = 11$. Is there more than one possible solution? Explain your reasoning.



THE “BIG IDEAS” OF PATTERNING AND ALGEBRA

All learning, especially new learning, should be embedded in well-chosen contexts for learning – that is, contexts that are broad enough to allow students to investigate initial understandings, identify and develop relevant supporting skills, and gain experience with varied and interesting applications of the new knowledge. Such rich contexts for learning open the door for students to see the “big ideas”, or key principles, of mathematics, such as pattern or relationship.

(Ontario Ministry of Education, 2005, p. 25)

About Big Ideas

Ginsburg, who has extensively studied young children doing mathematics, suggests that, although “mathematics is big”, children’s minds are bigger (2002, p. 13). He argues that “children possess greater competence and interest in mathematics than we ordinarily recognize”, and we should aim to develop a curriculum for them in which they are challenged to understand big mathematical ideas and have opportunities to “achieve the fulfilment and enjoyment of their intellectual interest” (p. 7).

In developing a mathematics program, it is important to concentrate on major mathematical themes, or “big ideas”, and the important knowledge and skills that relate to those big ideas. Programs that are organized around big ideas and focus on learning through problem solving provide cohesive learning opportunities that allow students to explore mathematical concepts in depth. An emphasis on big ideas contributes to the main goal of mathematics instruction – to help students gain a deeper understanding of mathematical concepts.

Teaching and Learning Mathematics: The Report of the Expert Panel on Mathematics in Grades 4 to 6 in Ontario, 2004 states that “When students construct a big idea, it is big because they make connections that allow them to use mathematics more effectively and powerfully. The big ideas are also critical leaps for students who are developing mathematical concepts and abilities” (p. 19).

Students are better able to see the connections in mathematics – and thus to learn mathematics – when it is organized in big, coherent “chunks”. In organizing a mathematics program, teachers should concentrate on the big ideas in mathematics and view the expectations in the curriculum policy documents for Grades 4 to 6 as being clustered around those big ideas.

The clustering of expectations around big ideas provides a focus for student learning and for teacher professional development in mathematics. Teachers will find that investigating and discussing effective teaching strategies for a big idea is much more valuable than trying to determine specific strategies and approaches to help students achieve individual expectations. In fact, using big ideas as a focus helps teachers to see that the concepts presented in the curriculum expectations should not be taught as isolated bits of information but rather as a network of interrelated concepts.

In building a program, teachers need a sound understanding of the key mathematical concepts for their students’ grade level, as well as an understanding of how those concepts connect with students’ prior and future learning (Ma, 1999). Such knowledge includes an understanding of the “conceptual structure and basic attitudes of mathematics inherent in the elementary curriculum” (p. xxiv), as well as an understanding of how best to teach the concepts to children. Concentrating on developing this knowledge will enhance effective teaching and provide teachers with the tools to differentiate instruction.

Focusing on the big ideas provides teachers with a global view of the concepts represented in the strand. The big ideas also act as a “lens” for:

- making instructional decisions (e.g., choosing an emphasis for a lesson or set of lessons);
- identifying prior learning;
- looking at students’ thinking and understanding in relation to the mathematical concepts addressed in the curriculum (e.g., making note of the ways in which a student solves a division problem);
- collecting observations and making anecdotal records;
- providing feedback to students;
- determining next steps;
- communicating concepts and providing feedback on students’ achievement to parents¹ (e.g., in report card comments).

Focusing on the big ideas also means that teachers use strategies for advancing all students’ mathematical thinking (Fraivillig, 2001) by:

- eliciting a variety of student solution methods through appropriate prompts, collaborative learning, and a positive, supportive classroom environment;

1. In this document, *parent(s)* refers to parent(s) and guardian(s).

- helping students develop conceptual understanding by attending to relationships among concepts;
- extending students' mathematical thinking by (a) encouraging them to try alternative ways of finding solutions and to generalize and (b) setting high standards of mathematical performance for all students.

In addition, a big-ideas focus means that teachers use strategies that help students learn to represent their own mathematical ideas using concrete materials, pictures, diagrams, mathematical notation, and language.

About the Teaching and Learning of Patterning and Algebra

Algebra is a language used for representing and exploring mathematical relationships. The view of algebra expressed in current curriculum documents emphasizes multiple representations of relationships between quantities, and stresses the importance of focusing student attention on the mathematical analysis of change in these relationships (Ontario Ministry of Education, 2005; NCTM, 2000). It is the relationships component of algebra that gives purpose and meaning to the language of algebra. Without a focus on relationships, the language of algebra loses its richness and is reduced to a set of grammatical rules and structures.

Some research indicates that the teaching and learning of algebra typically focuses on symbolic algebra over other representations (Kieran, 1992; Borba & Confrey, 1996; Kieran & Sfard, 1999, Nathan & Koedinger, 2000). Consequently, though students learn to manipulate algebraic expressions, they do not seem able to use them as tools for meaningful mathematical communication (Kieran & Sfard, 1999). The majority of students do not acquire any real sense of algebra and, early on in their learning of the subject, give up trying to understand algebra and resort to memorizing rules and procedures (Kieran, 1992, Kieran & Sfard, 1999).

Kieran & Sfard (1999) suggest that many students may find the rules of algebra arbitrary, because “all too often they are unable to see the mathematical objects to which these rules are supposed to refer” (p. 2). It has been suggested that students be given meaningful experiences in algebra learning, involving the exploration of multiple representations of concepts (Borba & Confrey, 1996, pp. 319–320; Kieran & Sfard, 1999, p. 3). It has also been suggested that the traditional approach to teaching algebra, which typically starts with symbolic representation and decontextualized manipulation and moves on to visual and graphical representation and problem-based contexts, should be reversed (Borba & Confrey, 1996, pp. 319–320). Graphs, which are often treated as a mere add-on to algebra, could become the foundation of algebra teaching and learning (Kieran & Sfard, 1999, p. 3).

Big Ideas and Tiered Instruction²

How students experience a “big idea”, and how “big” it becomes, depends greatly on how it is developed mathematically and pedagogically in the classroom. It is not enough to label a mathematical concept as a “big idea”. Big ideas must be coupled with a pedagogy that offers students opportunities to attend deeply to mathematical concepts and relationships.

Big ideas, and a pedagogy that supports student learning of big ideas, naturally provide opportunities for meeting the needs of students who are working at different levels of mathematical performance. The reason for this is that teaching around big ideas means teaching around ideas that incorporate a variety of levels of mathematical sophistication. For example, consider the problem of *finding all the different rectangular areas that can be enclosed by a fence of 24 m* and the tiers at which the problem can be approached or extended:

- **Tier 1:** Using square tiles or grid paper, students construct various areas whose perimeter is 24. Students record length, width, area, and perimeter in a table of values and look for patterns.
- **Tier 2:** Students also use the table of values to create ordered pairs based on length and area. They plot these ordered pairs on a graph and investigate relationships. For example, they might investigate which dimensions result in the greatest area, and consider whether their solution can be generalized.
- **Tier 3:** Students might also work on extensions of the problem. They could be asked, for example: “What if we wanted to create a rectangular pen whose area is 36 m^2 ? How many different rectangular shapes are possible? Is there a pattern? Which dimensions result in the least perimeter? Can we generalize the solution?”

Finding all the different rectangular areas that can be enclosed by a fence of 24 m is a variation of a sample problem that appears in a Grade 4 expectation within the Measurement strand of *The Ontario Curriculum, Grades 1–8: Mathematics, 2005* (p. 70). It is a problem that can be used to address Patterning and Algebra expectations in the junior grades (patterning and generalization). It addresses Data Management expectations (collecting and recording data that result from the various rectangles drawn). It also uses a context from the Geometry and Spatial Sense strand (quadrilaterals and their properties).

It should be noted that the problem of *finding all the different rectangular areas that can be enclosed by a fence of 24 m* is used as well at the Grade 10 level in the study of quadratic functions and at the Grade 12 level for the study of calculus concepts.

Big ideas, big problems, and a pedagogy that supports them at the classroom level provide opportunities for students to engage with *the same mathematical situation* at different levels of sophistication.

2. A tiered approach to instruction is suggested in *Education for All – The Report of the Expert Panel on Literacy and Numeracy Instruction for Students With Special Education Needs, Kindergarten to Grade 6, 2005* (pp. 60, 120, 123).

The Big Ideas of Patterning and Algebra in Grades 4 to 6

The goal of teaching and learning mathematics through big ideas is an integral component of *The Ontario Curriculum, Grades 1–8: Mathematics, 2005*. In each of the strands and in each of the grades, specific expectations have been organized around big ideas in mathematics.

In the case of the Patterning and Algebra strand in the junior grades, the big ideas are:

- patterns and relationships
- variables, expressions, and equations

The tables on the following pages show how the expectations for each of these big ideas progress through the junior grades.

The sections that follow offer teachers strategies and content knowledge to address these expectations in the junior grades, while helping students develop an understanding of patterning and algebra. Teachers can facilitate this understanding by helping students to:

- investigate patterning and algebra problems in real-life settings, and learn to calculate any term in a pattern when given the term number;
- extend their knowledge of generating patterns that involve addition, subtraction, multiplication, or division, as well as those involving reflections, translations, or rotations;
- investigate problems involving missing numbers and develop an early sense of variable;
- extend their understanding of equality of expressions using multiplication or division in equations with unknown quantities on both sides.

Curriculum Expectations Related to Patterns and Relationships, Grades 4, 5, and 6

By the end of Grade 4, students will:

Overall Expectations

- describe, extend, and create a variety of numeric and geometric patterns, make predictions related to the patterns, and investigate repeating patterns involving reflections;

Specific Expectations

- extend, describe, and create repeating, growing, and shrinking number patterns;
- connect each term in a growing or shrinking pattern with its term number, and record the patterns in a table of values that shows the term number and the term;
- create a number pattern involving addition, subtraction, or multiplication, given a pattern rule expressed in words;
- make predictions related to repeating geometric and numeric patterns;
- extend and create repeating patterns that result from reflections, through investigation using a variety of tools.

By the end of Grade 5, students will:

Overall Expectations

- determine, through investigation using a table of values, relationships in growing and shrinking patterns, and investigate repeating patterns involving translations;

Specific Expectations

- create, identify, and extend numeric and geometric patterns, using a variety of tools;
- build a model to represent a number pattern presented in a table of values that shows the term number and the term;
- make a table of values for a pattern that is generated by adding or subtracting a number (i.e., a constant) to get the next term, or by multiplying or dividing by a constant to get the next term, given either the sequence or the pattern rule in words;
- make predictions related to growing and shrinking geometric and numeric patterns;
- extend and create repeating patterns that result from translations, through investigation using a variety of tools.

By the end of Grade 6, students will:

Overall Expectations

- describe and represent relationships in growing and shrinking patterns (where the terms are whole numbers), and investigate repeating patterns involving rotations;

Specific Expectations

- identify geometric patterns, through investigation using concrete materials or drawings, and represent them numerically;
- make tables of values for growing patterns, given pattern rules in words, then list the ordered pairs (with the first coordinate representing the term number and the second coordinate representing the term) and plot the points in the first quadrant, using a variety of tools;
- determine the term number of a given term in a growing pattern that is represented by a pattern rule in words, a table of values, or a graph;
- describe pattern rules (in words) that generate patterns by adding or subtracting a constant, or multiplying or dividing by a constant, to get the next term, then distinguish such pattern rules from pattern rules, given in words, that describe the general term by referring to the term number;
- determine a term, given its term number, by extending growing and shrinking patterns that are generated by adding or subtracting a constant, or multiplying or dividing by a constant, to get the next term;
- extend and create repeating patterns that result from rotations, through investigation using a variety of tools.

Curriculum Expectations Related to Variables, Expressions, and Equations, Grades 4, 5, and 6

By the end of Grade 4, students will:	By the end of Grade 5, students will:	By the end of Grade 6, students will:
<p>Overall Expectations</p> <ul style="list-style-type: none"> demonstrate an understanding of equality between pairs of expressions, using addition, subtraction, and multiplication; <p>Specific Expectations</p> <ul style="list-style-type: none"> determine, through investigation, the inverse relationship between multiplication and division; determine the missing number in equations involving multiplication of one- and two-digit numbers, using a variety of tools and strategies; identify, through investigation, and use the commutative property of multiplication to facilitate computation with whole numbers; identify, through investigation, and use the distributive property of multiplication over addition to facilitate computation with whole numbers. 	<p>Overall Expectations</p> <ul style="list-style-type: none"> demonstrate, through investigation, an understanding of the use of variables in equations; <p>Specific Expectations</p> <ul style="list-style-type: none"> demonstrate, through investigation, an understanding of variables as changing quantities, given equations with letters or other symbols that describe relationships involving simple rates; demonstrate, through investigation, an understanding of variables as unknown quantities represented by a letter or other symbol; determine the missing number in equations involving addition, subtraction, multiplication, or division and one- or two-digit numbers, using a variety of tools and strategies. 	<p>Overall Expectations</p> <ul style="list-style-type: none"> use variables in simple algebraic expressions and equations to describe relationships; <p>Specific Expectations</p> <ul style="list-style-type: none"> demonstrate an understanding of different ways in which variables are used; identify, through investigation, the quantities in an equation that vary and those that remain constant; solve problems that use two or three symbols or letters as variables to represent different unknown quantities; determine the solution to a simple equation with one variable, through investigation using a variety of tools and strategies.



PATTERNS AND RELATIONSHIPS

Overview

Teaching and Learning Mathematics: The Report of the Expert Panel on Mathematics in Grades 4 to 6 in Ontario, 2004 (p. 1) states:

The junior years are an important time of transition and growth in students' mathematical thinking. [...] Junior students investigate increasingly complex ideas, building on their capacity to deal with more formal concepts. For example, students learn to generalize patterns without having to draw each stage and record each term.

Students enter the junior grades with experience in and knowledge in recognizing, extending, and creating patterns in various contexts. As they move through the junior grades, they consolidate the following skills:

- generating patterns;
- predicting terms in a pattern;
- determining any term given the term number;
- determining a pattern rule for a growing or shrinking pattern;
- describing pattern rules in words;
- describing patterns using tables of values, ordered pairs, and graphs;
- distinguishing between a term in a growing pattern and its term number.

They also develop a more abstract understanding of *pattern* by exploring ways of describing the *general term* of a sequence. For example:

- For the sequence of even numbers (2, 4, 6, 8 ...), any *term* can be determined by doubling the *term number*. For example, the 3rd term in the sequence of even numbers is 6. The term number of this term is 3, since it is the 3rd number in the sequence. If we double the term number we get $2 \times 3 = 6$. Likewise, the 5th term in the sequence is $2 \times 5 = 10$. This relationship between term and term number can be expressed using an algebraic expression: the general term in the sequence of even numbers is given as 2 times the term number.

- For the sequence of odd numbers (1, 3, 5, 7 ...), any *term* can be determined by doubling the *term number* and then subtracting 1. Therefore, the 5th term in this sequence is $2 \times 5 - 1 = 10 - 1 = 9$. Using the variable n , where n represents the term number, the general term can be expressed as 2 times the term number minus 1, or $2 \times n - 1$.

Teaching and Learning Mathematics: The Report of the Expert Panel on Mathematics in Grades 4 to 6 in Ontario, 2004 (p. 19) states:

The junior years are an important time of transition and growth in students' mathematical thinking. [...] Junior students investigate increasingly complex ideas, building on their capacity to deal with more formal concepts. For example, students learn to generalize patterns without having to draw each stage and record each term.

Example: Growing Patterns

Let's see how the "big idea" of "patterns and relationships" might be developed through a student exploration of the growing L-pattern problem (see Figure 1). This problem is listed in the Patterns and Relationships expectations for Grade 5 in *The Ontario Curriculum, Grades 1–8: Mathematics, 2005* (p. 83). You can see a video of a Grade 4 student working with this problem at <http://www.edu.uwo.ca/dmp/Lpatterns>.

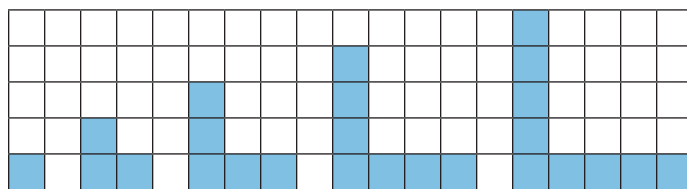


Figure 1: A growing pattern

The pattern shown in Figure 1 may be presented to students using a diagram or using concrete materials such as square tiles. The pattern may then be investigated (as in Method 1 below) so that students can develop an understanding of how it grows numerically as well as geometrically. The problem may also be extended to allow students to engage in higher-level problem solving (see Method 2 on p. 25).

Method 1: Understanding the Pattern

LEARNING TO GENERALIZE

How can we investigate the relationships in the pattern and develop a rule for finding the 20th stage without having to draw all the diagrams that precede it? As noted in *Teaching and Learning Mathematics: The Report of the Expert Panel on Mathematics in Grades 4 to 6 in Ontario, 2004* (p. 19), learning how to solve this type of problem is very important for students' mathematical development and for their understanding of patterning and algebra. Resolution of the problem involves them in mathematical generalization. Let's consider the problem of finding the 10th stage of the L pattern shown in Figure 1. One way to solve the

problem is to extend the pattern until we reach the 10th stage. However, such a method would not be very efficient if we wanted to find the 20th stage or the 100th stage. A more powerful solution involves looking for a more general way to understand the pattern.

We might hear someone say, when discussing the pattern, that the stage number matches the number of squares along each of the arms of the L pattern. In the 2nd stage, the arms of the L pattern are 2 squares long; in the 3rd stage, the arms are 3 squares long; in the 4th stage, the arms are 4 squares long; and so forth (as in Figure 2). Therefore, for the 100th stage the L pattern has 100 squares on each of the arms.

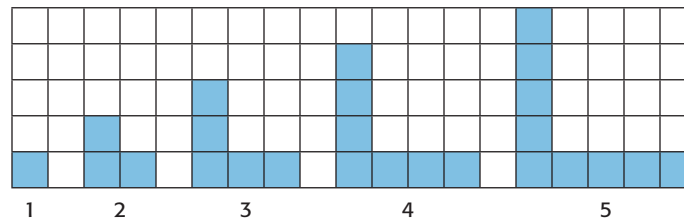


Figure 2: The stage number equals the number of squares on each arm of the L pattern

Blanton and Kaput (2003) suggest that teachers should prompt for and learn to recognize students' algebraic thinking, which can be encouraged by asking simple questions such as:

- Could you tell me what you were thinking?
- Did you solve this problem a different way?
- How do you know that this solution is correct?
- Does this method always work?

"These questions not only reveal students' thinking but also prompt students to justify, explain, and build arguments – processes that lie at the heart of algebraic reasoning" (p. 73).

CHARACTERISTICS OF STUDENT LEARNING AND INSTRUCTIONAL STRATEGIES

Students who understand the pattern in Figure 1 are able to:

- extend the pattern;
- describe in their own words how the pattern grows;
- make predictions about, say, the 10th stage of the pattern;
- describe a general rule for determining any stage of the pattern.

To develop such an understanding of the pattern, teachers use instructional strategies that help students answer the following questions:

- How does the pattern grow?
- What changes as the pattern grows?
- What stays the same as the pattern grows?

Method 2: Extending the Problem

It is the questions that drive mathematics. Solving problems and making up new ones is the essence of mathematical activity.

(Hersh, 1997, p. 18)

The skills identified in *Method 1: Understanding the Pattern* need to be practised with a variety of growing patterns and in a variety of contexts: numbers, diagrams, concrete materials, and real-life applications. Students also need to explore some growing patterns in a more in-depth fashion, in which the problem is extended by using a new context or a “what if” question. This goal is reflected in the mathematical process expectations listed for junior students in *The Ontario Curriculum, Grades 1–8: Mathematics, 2005* (p. 77): students will “pose and solve problems and conduct investigations to help deepen their mathematical understanding”. *The Report of the Expert Panel on Mathematics in Grades 4 to 6 in Ontario, 2004* (p. 7) states that “a variety of problem-solving experiences and a balanced array of pedagogical approaches” are necessary for effective mathematics instruction in the junior grades. In fact, balance is an essential aspect of an effective mathematics program (Kilpatrick, Swafford, & Findell, 2001).

The extension examples below (starting on p. 26) aim to give a sense of the richness – the “bigness” – of mathematical ideas. Knowing the mathematical connections discussed in the extensions to the L-pattern problem below will help teachers plan investigations that engage students with big mathematical ideas and will also help them be flexible and responsive to mathematical questions and directions initiated by students.

Extensions engage students’ minds and give them a sense of the beauty of patterning and algebra, and of mathematics in general. For example, to find the sum of the first 10 odd numbers, we could use paper and pencil or a calculator and conclude that $1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19 = 100$. But Figure 3 shows the beauty of the visual demonstration of finding the sum of odd numbers. When we discover that such sums of odd numbers always result in square numbers, we not only have the solution to the problem of finding the sum of the first 10 odd numbers (10×10) but also have a general solution for finding the sum of the first n odd numbers ($n \times n$).

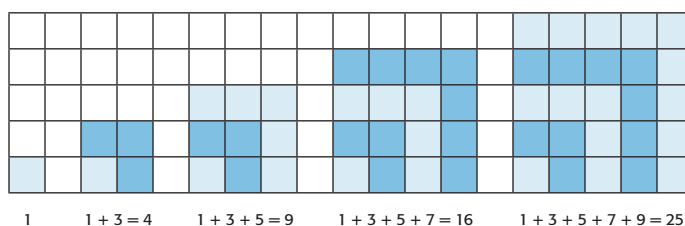


Figure 3: Demonstrating that the sum of the first n odd numbers is $n \times n$

Exploring representations and connections such as these, in the context of growing patterns, makes the mathematics more interesting to students. Learning to attend to a variety of mathematical representations and connections also helps make students better mathematical thinkers.

CHARACTERISTICS OF STUDENT LEARNING AND INSTRUCTIONAL STRATEGIES

Extending a problem in mathematically more sophisticated ways requires a change in the role most teachers play in the classroom. Teachers must be flexible and responsive to the directions of mathematical exploration, both anticipated and unanticipated, that are bound to emerge in a classroom setting. In addition, the teacher needs to use more elaborate instructional strategies in which:

- students are encouraged to pose and explore “what if” questions related to the pattern;
- students are encouraged to model the problem situations, using a variety of representations: numbers, diagrams, concrete materials, computer models, and real-life applications;
- students are encouraged to make connections with patterns explored previously and with concepts learned in other strands;

Students who develop problem-solving skills that enable them to explore extensions to patterning problems have the following learning characteristics:

- they can pose “what if” questions to extend problems in new mathematical directions;
- they are willing to persevere in their mathematical thinking and solve mathematical problems;
- they can work cooperatively and constructively with others;
- they use a variety of methods to test hypotheses and generalizations.

The extensions below are intended as notes to the teacher and are not fully developed pedagogically. The aim is to give the teacher glimpses into the mathematical extensions that surround and connect to big ideas. The teacher needs to be aware of such extensions so as to design classroom investigations that offer opportunities for students to make connections and explore related ideas.

EXTENSION: HOW MANY SQUARE TILES ARE NEEDED?

The growing-pattern problem discussed in Method 1 might be extended as adapted from <http://www.edu.uwo.ca/dmp/Lpatterns>:

When we use square tiles to build the growing pattern shown in Figure 4, we might wonder how many square tiles would be needed to build the first 10 stages of the pattern.

Such a question changes the problem from finding stages in a sequence to finding the sum of the terms in a sequence.

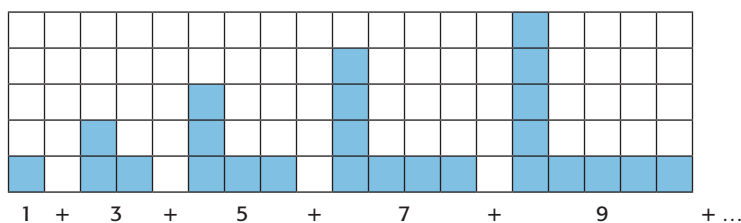


Figure 4: Finding the number of tiles needed

As we count the number of tiles needed to construct each stage of the sequence, we notice that we end up with a sum of odd numbers. Noticing that there is a connection between the number of tiles and the odd numbers helps create a link between a geometric and a numeric representation of the problem.

EXTENSION: WHAT IS THE SUM OF THE FIRST 10 ODD NUMBERS?

From Figure 4 we can see that to answer the question “How many square tiles would we need to build the first 10 stages of the pattern?” we need to find the sum of the first 10 odd numbers. One way to do this is by using a calculator:

$$1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19 = 100$$

However, this method would be cumbersome even with a calculator if we wanted to solve the problem for 50 or 100 stages. Let’s explore a mathematically more powerful solution method.

Looking at the odd numbers from 1 to 19 we might notice that $1 + 19 = 20$, $3 + 17 = 20$, $5 + 15 = 20$, and so forth. There are 5 pairs that, when added together, give us 20; therefore, the sum is $5 \times 20 = 100$.

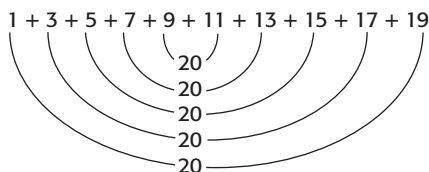


Figure 5: Finding the sum of the first 10 odd numbers

EXTENSION: IS THE ANSWER ALWAYS A SQUARE NUMBER?

It is interesting that the sum of the first 10 odd numbers is 100, which is a square number ($100 = 10 \times 10$). We can also observe in Figure 3 that other such sums result in square numbers. We might wonder whether this is a general rule. If we try a few more cases to test the rule we discover that it continues to hold true.

- The sum of the first 20 odd numbers is $20 \times 20 (= 400)$;
- the sum of the first 50 odd numbers is $50 \times 50 (= 2500)$;
- the sum of the first 1000 odd numbers is $1000 \times 1000 (= 1\,000\,000)$;
- and so on.

EXTENSION: A GEOMETRIC PROOF?

We started with diagrams and concrete materials and moved on to numbers to represent the growing pattern. If we go back to using diagrams and concrete materials, we might notice that the L patterns of the sequence fit into one another to form a growing-square pattern, providing visual proof that the sum of the first n odd numbers is $n \times n$ (as in Figure 6). The square thus formed has a side length of 10 units, so its area is 100 square units (or square tiles).

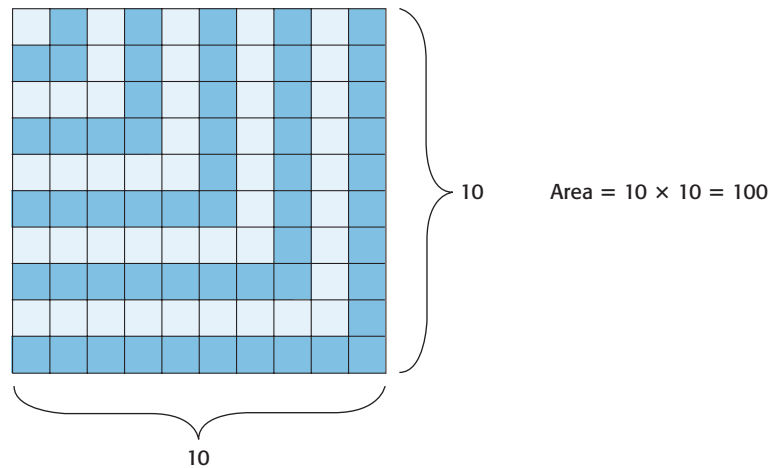


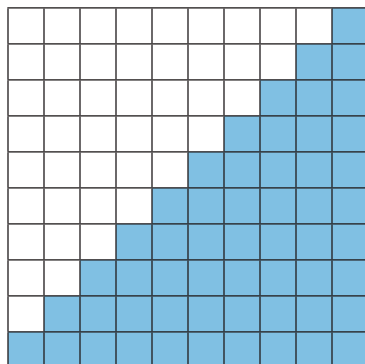
Figure 6: A visual representation of the sum of the first 10 odd numbers

When we look at the pattern in Figure 6 we experience a sense of aesthetic satisfaction, a sense of mathematical pleasure. The image, with its fitted-together Ls, draws our attention; it says, “look at me”. We sense the pleasure of seeing the connection between the odd numbers, their geometric representations as Ls, and the visual proof that the sum of the first 10 odd numbers is 10×10 .

Experiences such as these help students develop an understanding and an appreciation of mathematics.

EXTENSION: WHAT IS THE SUM OF THE FIRST 10 NATURAL NUMBERS?

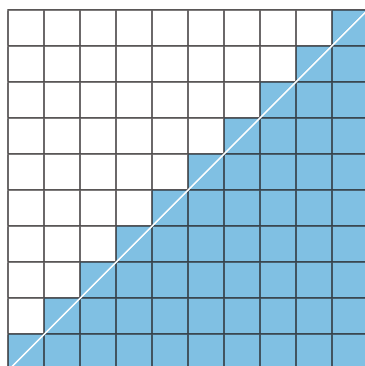
If this type of visual proof works for finding the sum of odd numbers, students might wonder whether there is a visual proof for finding the sum of natural numbers (that is, $1 + 2 + 3 + 4 + 5 + 6 + \dots$). Figure 7 shows a visual representation of the sum of the first 10 natural numbers. Can you see a geometric way of finding the sum of these numbers?



$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$$

Figure 7: A visual representation of the sum of the first 10 natural numbers

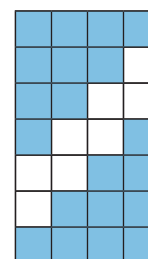
By drawing the line shown in Figure 8, students can see that the sum (or the number of tiles needed to represent the first 10 natural numbers) is a little more than half the area of the 10×10 square.



$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$$

Figure 8: A second visual representation of the sum of the first 10 natural numbers

Can you develop an expression or rule for finding the sum of the first 10 natural numbers? Consider the case of the sum $1 + 2 + 3 + 4$. Notice that this sum is twice represented as an area in the diagram on the right. If you slide the two areas together you get a rectangle with width 4 and length 5. The area of the rectangle is 4×5 . The area of half of the rectangle, which is also the sum $1 + 2 + 3 + 4$, is $(4 \times 5)/2$. Can you see how this method can be used to find the sum of the first 10 natural numbers?



EXTENSION: MATHEMATICAL LITERATURE

Whitin and Wilde (1992) identify the many benefits of using children's literature in mathematics teaching. Children's literature:

- provides a meaningful context for mathematics;
- celebrates mathematics as language;
- demonstrates that mathematics develops out of human experience;
- addresses humanistic, affective elements of mathematics;
- integrates mathematics into other curriculum areas;
- restores an aesthetic dimension to mathematical learning;
- provides a meaningful context for posing problems.

Mathematical literature can be a starting point for engaging and extending students' algebraic thinking. Books such as *One Grain of Rice* (Demi, 1997) and *The King's Chessboard* (Birch & Grebu, 1988) offer students opportunities for exploring doubling patterns. For example, they are asked to consider this problem: If 1 grain of rice is placed on the 1st square of a chessboard, 2 grains of rice on the 2nd, 4 on the 3rd, 8 on the 4th, and so on, how many grains would be needed to complete the pattern? The problem is related to the following puzzle: Would you prefer to receive \$100 000 per day for 30 days, or 1 cent on the 1st day, 2 cents on the 2nd, 4 cents on the 3rd, 8 on the 4th, and so on, for 30 days? The first pattern results in \$3 000 000 while the second pattern results in 1 073 741 823 cents or \$10 737 418.23. Another example: in *Anno's Magic Seeds* (Anno, 1995) a more complex growing pattern is embedded in the story of a farmer and his magic seeds. Literature builds the context for problem solving, providing motivation and purpose for approaching the mathematics.

VARIABLES, EXPRESSIONS, AND EQUATIONS

Overview

Students enter the junior grades with experience in finding *missing numbers* in simple equations and in using algebraic thinking. They have explored the concept of *equality* in the context of number operations (for example, using the associative property to see $17 + 16$ as $17 + 3 + 13$).

As they move through the junior grades, students use the concept of equality inherent in the distributive, associative, and commutative properties to better understand operations with numbers. For example, they use the commutative property to express $15 \times 7 \times 2$ as $15 \times 2 \times 7$. And they use the distributive property to express 12×18 as $10 \times 18 + 2 \times 18$ or as $10 \times 10 + 10 \times 8 + 2 \times 10 + 2 \times 8$, and in the process learn to visualize multiplication through an area model (see Figure 9). The development of this understanding typically starts by exploring situations such as the diagram shown in Figure 9 and then continues by making connections to the distributive property.

10	10×10	10×8
2	2×10	2×8
	10	8

Figure 9. Using the distributive property

Students in the junior grades also develop a more sophisticated understanding of *expressions*, *variables*, and *equations*.

- They solve more complex missing-number problems (in Grade 4, for instance, students solve problems such as $2 \times _ = 24$).
- They expand their understanding of “variable” from simply seeing it as an unknown (as in $a + 3 = 10$) to also seeing it as a varying quantity, and they develop an early sense of covariation (as in $a + b = 10$, where a change in the value of one of the variables affects the value of the other variable).

Example: Missing Numbers³

One of the skills that students extend in the junior grades is that of finding missing numbers. For example, they learn about and practise finding missing numbers in equations such as:

- $_ + 3 = 12$
- $\square \div 3 = 4$
- $15 - y = 7$
- $3 \times _ = 21$

Although it is important for students to practise and develop the skill of finding missing numbers in equations, the concept on its own is not a big idea in junior-grade mathematics, where variables are portrayed as only representing an unknown. One way to make the learning of this skill more meaningful mathematically is to relate it to bigger ideas about equations and variables, where variables represent changing quantities.

For example, suppose we start with the equation $_ + _ = 10$.

- Working in pairs, students roll a number cube for the first missing number and then calculate the second missing number. Let’s say a 4 is rolled; the equation becomes $4 + _ = 10$. In this case, the second missing number is calculated to be 6. Students are asked to record each solution as a number sentence, without repeating identical solutions. That is, if a 4 is rolled a second time, the corresponding equation $4 + _ = 10$ is not recorded again.
- Students continue rolling the number cube and finding the missing numbers until all possibilities are exhausted. As the number cube has six sides, with six different numbers, students should conclude that there are six different pairs of numbers.

³ This activity is adapted from Gadanidis, G. (2004). The pleasure of attention and insight. *Mathematics Teaching* 186 (1), pp. 10–13.

- Their six number sentences might look like the following:

$$\begin{aligned}4 + 6 &= 10 \\2 + 8 &= 10 \\5 + 5 &= 10 \\3 + 7 &= 10 \\1 + 9 &= 10 \\6 + 4 &= 10\end{aligned}$$

Figure 10. Listing all the possibilities

- Students then rewrite their equations in an order that demonstrates a pattern.

$$\begin{aligned}1 + 9 &= 10 \\2 + 8 &= 10 \\3 + 7 &= 10 \\4 + 6 &= 10 \\5 + 5 &= 10 \\6 + 4 &= 10\end{aligned}$$

Figure 11. Listing all the possibilities, using patterning

- Such a representation allows students to observe the covariation pattern between the number rolled and the number calculated. As one number increases, the other decreases, and vice versa.

The above activity helps students develop an understanding of covariation (see Method 1 below for more examples of activities that address this concept). The activity may also be extended to help students see a more sophisticated view of algebra and variables (see Method 2 on p. 35).

Method 1: Understanding Variables

In the junior grades, it is important that the teacher seek ways to help students investigate and experience covariation in a variety of settings. For example, when students are studying area and perimeter, the teacher can pose problems such as the two suggested below. These problems address the following Grade 5 Measurement expectation in a manner that integrates algebraic thinking: “create, through investigation using a variety of tools (e.g., pattern blocks, geoboard, grid paper) and strategies, two-dimensional shapes with the same perimeter or the same area (Sample problem: Using dot paper, how many different rectangles can you draw with a perimeter of 12 units? with an area of 12 square units?)”.

HOW MUCH FENCING IS NEEDED FOR A GIVEN AREA?

Problem 1: Suppose we wanted to create a rectangular pen for a dog, with an area of 64 m^2 . What length of fence would we need? Do the dimensions of the pen matter?

This problem engages students in finding missing numbers in a situation where two variables covary. This covariation can be expressed as $_ \times _ = 64$ or $\ell \times w = 64$. Notice that there are many possible solutions (see Figure 12).

ℓ (Length)	w (Width)	Area	Fencing (Perimeter)
1 m	64 m	64 m ²	130 m
2 m	32 m	64 m ²	68 m
4 m	16 m	64 m ²	40 m
8 m	8 m	64 m ²	32 m
10 m	6.4 m	64 m ²	32.8 m
16 m	4 m	64 m ²	40 m
...	...	64 m ²	...

Figure 12. What is the least amount of fencing needed?

HOW MUCH AREA IS ENCLOSED FOR A GIVEN AMOUNT OF FENCING?

Problem 2: Suppose we have 32 m of fencing. If we wanted to use this fencing to create a rectangular pen, what would be the area of the pen? Which dimensions would result in the biggest area?

This problem also engages students in finding missing numbers in a situation where two variables covary. The covariation can be expressed as $2 \times _ + 2 \times _ = 32$ or $2 \times \ell + 2 \times w = 32$ (see Figure 13).

ℓ (Length)	w (Width)	Fencing (Perimeter)	Area
1 m	15 m	32 m	15 m ²
2 m	14 m	32 m	28 m ²
3 m	13 m	32 m	39 m ²
4 m	12 m	32 m	48 m ²
5 m	11 m	32 m	55 m ²
6 m	10 m	32 m	60 m ²
...	...	32 m	...

Figure 13. What is the largest rectangular area we can create?

Investigating such problems enriches students' algebraic understanding and also helps extend students' understanding of the relationships between area and perimeter. Such problems also offer a contextualized opportunity for students to practise finding missing numbers and using the area and perimeter formulas for rectangles. This problem is further elaborated in the big ideas section of *Measurement* (pp. 38–39).

CHARACTERISTICS OF STUDENT LEARNING AND INSTRUCTIONAL STRATEGIES

Junior students who understand variables:

- can see a variable as an unknown (as in $_ + 3 = 12$, or $x - 5 = 9$, or $2x + 1 = 15$);
- recognize and use variables in various forms (as in $_ + 3 = 12$, or $x - 5 = 9$, or $2x + 1 = 15$);
- recognize variables in formulas (as in $A = \ell \times w$, or $P = 2 \times \ell + 2 \times w$);
- can work with missing number equations where two variables covary (as in $x + y = 10$).

To help students develop an understanding of variables, teachers must provide them with opportunities to experience the various facets of variables.

Method 2: A More Sophisticated View of Algebra and Variables

For students to develop a more sophisticated view of algebra and variables, they need opportunities to investigate the relationships among an equation and its table of values (and the ordered pairs that can be created from the table of values).

EXTENSION: ORDERED PAIRS AND GRAPHS

Let's use the $_ + _ = 10$ missing-number activity from page 32 to illustrate this method.

The list of number sentences in Figure 11 offers a meaningful setting for introducing the concept of an *ordered pair* to junior students (Ontario Ministry of Education, 2005, p. 93).

Let's suppose we roll a number cube to get the first missing number in $_ + _ = 10$ and then calculate the second missing number. The pairs of numbers we then have are ordered: the first number is always the number rolled and the second number is always the number calculated. Students could write the corresponding ordered pair beside each equation:

Equation	Ordered Pair
$1 + 9 = 10$	(1, 9)
$2 + 8 = 10$	(2, 8)
$3 + 7 = 10$	(3, 7)
$4 + 6 = 10$	(4, 6)
$5 + 5 = 10$	(5, 5)
$6 + 4 = 10$	(6, 4)

Figure 14. Writing ordered pairs

Using a coordinate grid, students plot the ordered pairs (see Figure 15). The first number in the ordered pair (which could be called R , for “rolled”) represents the horizontal coordinate, while the second number (which could be called C , for “calculated”) represents the vertical coordinate.

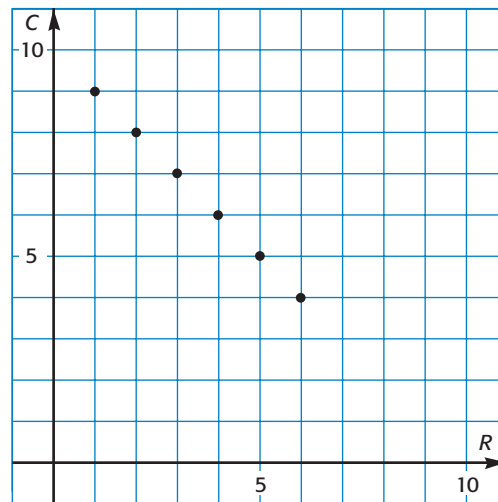


Figure 15. Plotting ordered pairs

The completed graph catches students by surprise: they see that the plotted ordered pairs line up. Such surprises are opportunities to explore further. Ask:

- What if we changed the 10 in $_ + _ = 10$ to a different number?
- What would happen to the graph?

Let’s consider the case of $_ + _ = 8$.

- Students could predict the shape of the graph of $_ + _ = 8$.
- Then they could test their prediction by repeating the steps they performed for $_ + _ = 10$, plotting their graph on the same grid, so that the two graphs can easily be compared (see Figure 16).

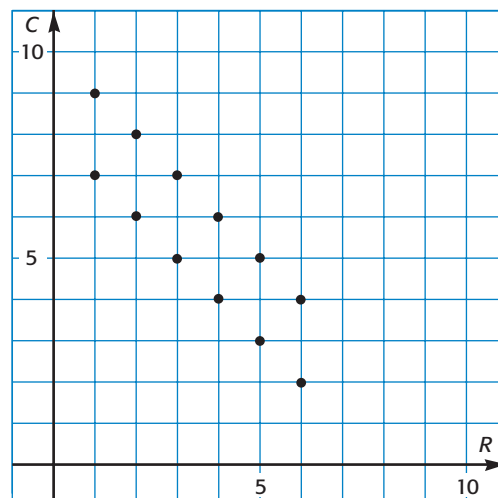


Figure 16. Plotting more ordered pairs

The missing-number equations $_ + _ = 10$ and $_ + _ = 8$ offer the opportunity to introduce and discuss the concept of *variable*, which is typically represented by a letter. The letters used for the variables could be arbitrary (e.g., $x + y = 10$) or meaningful, such as:

- $R + C = 10$ (where R is the number rolled and C is the number calculated);
- $H + V = 10$ (where H is the horizontal coordinate and V is the vertical coordinate).

Such equations offer the opportunity to attend to the difference between a variable as a place holder and a variable as a changing quantity.

In a typical missing-number exercise (e.g., $b + 3 = 12$), the variable is a placeholder for the missing number. In an equation such as $x + y = 10$, there is a situation of covariation, where not only is a variable expected to change, but the change in the value of one of the variables also affects the value of the second variable.

The equations are also useful because they can serve as labels for the graph (see Figure 17).

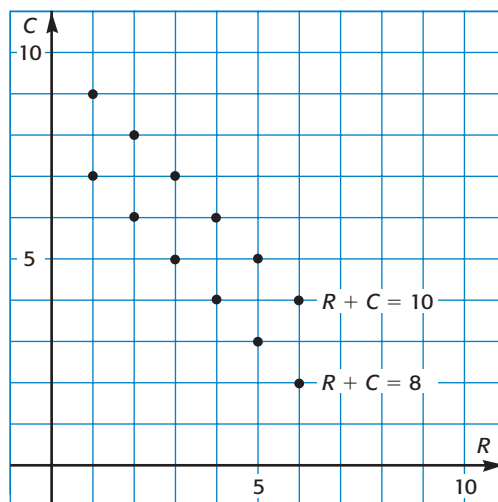


Figure 17. Labelling the graph

Such experiences allow students to see a graphical representation of their work with missing-number equations, and offer opportunities to wonder and explore and use their mathematical imaginations. To encourage such thinking, ask:

- How can we predict the graph of $R + C = 12$?
- How can we write an equation whose graph line points in a different direction?
- Can we write an equation whose graph line is curved?

EXTENSION: GRAPHS OF AREA AND PERIMETER

The area and perimeter activities discussed in Method 1 (and shown in Figures 12 and 13) can also serve as settings in which students graph ordered pairs and see them come to life (by plotting length versus perimeter for Figure 12, and length versus area for Figure 13).

Investigating such problems deepens students' understanding of patterning and algebra relationships. Such problems also provide a contextualized opportunity for students to make connections between patterning and algebra concepts and concepts from other strands.

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Learning Activities

Introduction to the Learning Activities

The following learning activities for Grades 4, 5, and 6 provide teachers with instructional ideas that help students achieve some of the curriculum expectations related to patterning and algebra. The learning activities also support students in developing their understanding of the big ideas outlined in the first part of this guide.

The learning activities do not address all concepts and skills outlined in the curriculum document, nor do they address all the big ideas – one activity cannot fully address all concepts, skills, and big ideas. The learning activities demonstrate how teachers can introduce or extend mathematical concepts; however, students need multiple experiences with these concepts to develop a strong understanding.

Each learning activity is organized as follows:

OVERVIEW: A brief summary of the learning activity is provided.

BIG IDEA(S): The big ideas that are addressed in the learning activity are identified. The ways in which the learning activity addresses these big ideas are explained.

CURRICULUM EXPECTATIONS: The curriculum expectations are indicated for each learning activity.

ABOUT THE LEARNING ACTIVITY: This section provides guidance to teachers about the approximate time required for the main part of the learning activity, as well as the materials, mathematics language, and instructional groupings for the learning activity.

ABOUT THE MATH: Background information is provided about the mathematical concepts and skills addressed in the learning activity.

GETTING STARTED: This section provides the context for the learning activity, activates prior knowledge, and introduces the problem or task.

WORKING ON IT: In this part, students work on the mathematical task, often in small groups or with a partner. The teacher interacts with students by providing prompts and asking questions.

REFLECTING AND CONNECTING: This section usually includes a whole-class debriefing time that allows students to share strategies and the teacher to emphasize mathematical concepts.

TIERED INSTRUCTION: These are suggestions for ways to meet the needs of all learners in the classroom.

HOME CONNECTION: This section is addressed to parents or guardians, and includes a task for students to do at home that is connected to the mathematical focus of the learning activity.

ASSESSMENT: This section provides guidance for teachers on assessing students' understanding of mathematical concepts.

BLACKLINE MASTERS: These pages are referred to and used throughout the activities and learning connections.

Grade 4 Learning Activity

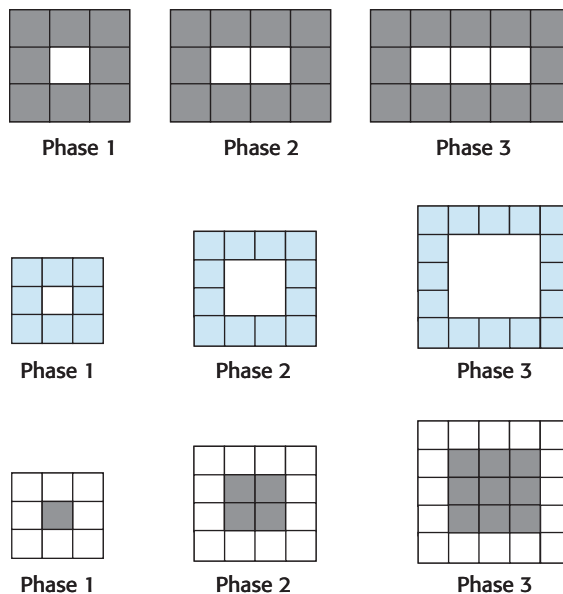
Picnic Partners

OVERVIEW

This learning activity can be used to introduce students to the exploration of growing patterns (sequences). The context is the seating around a picnic table.

Students learn about or review the use of T-charts and are introduced to the concept of pattern rules. By solving the problem, students will represent their thinking about patterns in a variety of ways, including concrete materials (manipulatives), numbers, words, tables, diagrams, and graphs. As students model and explain their representations of the patterns, they will build connections from concrete experiences toward generalizations of their findings using mathematical language.

Prior to this learning activity, students should have had some experience with extending simple number patterns, using charts to display data, using concrete materials to represent patterns, and representing simple geometric patterns with the aid of a number sequence, a number line, or a bar graph.



BIG IDEA

Patterns and relationships

CURRICULUM EXPECTATIONS

The learning activity addresses the following **specific expectations**.

Students will:

- extend, describe, and create repeating, growing, and shrinking number patterns (e.g., “I created the pattern 1, 3, 4, 6, 7, 9, I started at 1, then added 2, then added 1, then added 2, then added 1, and I kept repeating this.”);

- connect each term in a growing or shrinking pattern with its term number (e.g., in the sequence 1, 4, 7, 10, ..., the first term is 1, the second term is 4, the third term is 7, and so on), and record the patterns in a table of values that shows the term number and the term;
- create a number pattern involving addition, subtraction, or multiplication, given a pattern rule expressed in words (e.g., the pattern rule “start at 1 and multiply each term by 2 to get the next term” generates the sequence 1, 2, 4, 8, 16, 32, 64, ...);
- make predictions related to repeating geometric and numeric patterns.

These expectations contribute to the following **overall expectation**.

Students will:

- describe, extend, and create a variety of numeric and geometric patterns, make predictions related to the patterns, and investigate repeating patterns involving reflections.

TIME:
2 hours

ABOUT THE LEARNING ACTIVITY

MATERIALS

- interlocking cubes or two-coloured tiles (about 80 per pair of students)
- graph paper
- grid chart paper, markers
- calculators
- **PA.BLM4a.1: Space Station Challenge** (1 per student)
- **PA.BLM4a.2: Space Station Challenge Home Connection** (1 per student)

MATH LANGUAGE

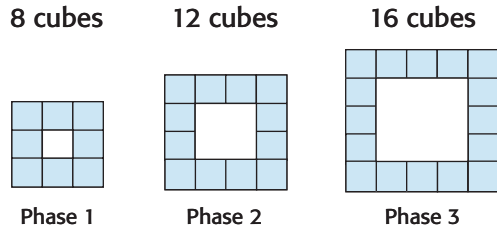
- growing pattern
- sequence
- border
- perimeter area
- T-chart

ABOUT THE MATH

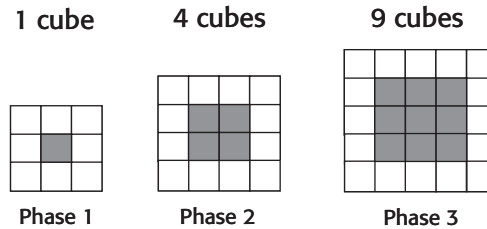
INSTRUCTIONAL GROUPING:
whole group,
pairs/small group

DIFFERENT GROWTH PATTERNS

Consider the difference between these two growth patterns. The first pattern grows when the same number is added to each term – that is, the pattern grows at a constant rate of 4. The second pattern grows when an increasing amount is added to each term – that is, the growth rate of the pattern is not constant.

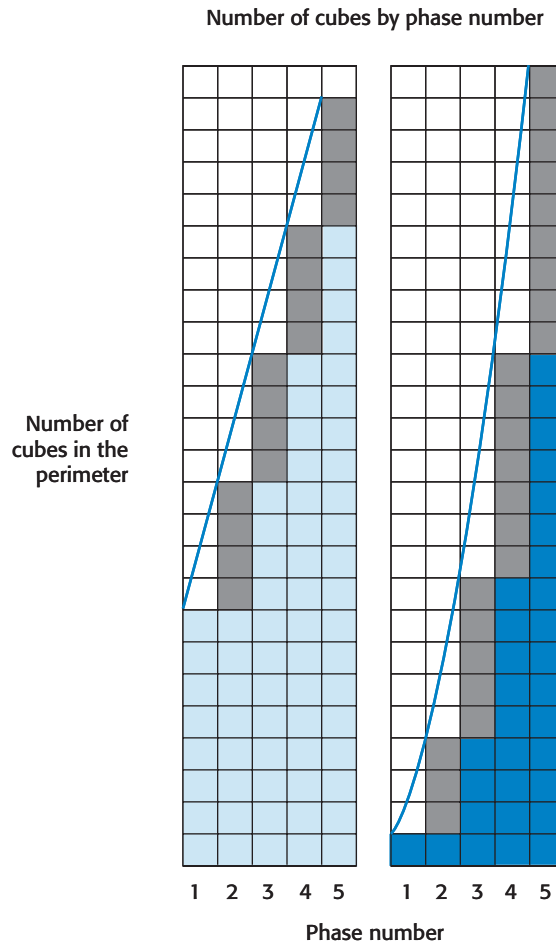
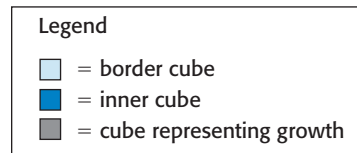


Same number added to each term



Increasing number added to each term

The bar graphs on the right show the number of shaded squares in the patterns. Notice that the bars that grow by adding the same number form a straight line pattern while the bars that grow by adding an increasing amount form a curve.



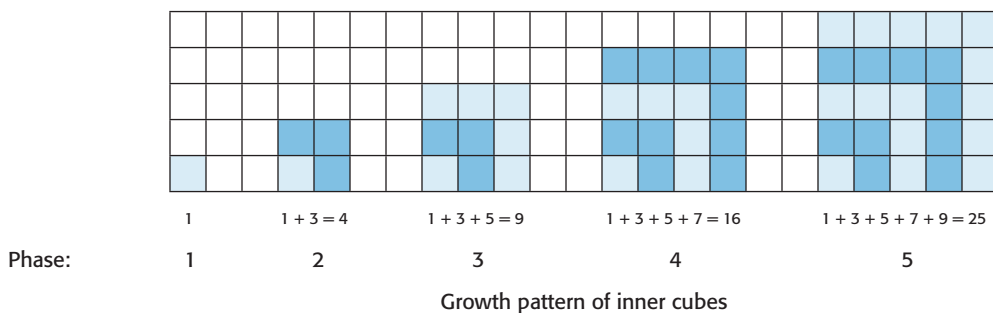
THE GROWTH PATTERNS

The table below shows how the two types of pattern grow.

Dimensions of the Phases	Number of Border Cubes	Border Growth	Number of Inner Cubes	Inner Growth
3×3	8	+ 4	1	+ 3
4×4	12	+ 4	4	+ 5
5×5	16	+ 4	9	+ 7
6×6	20	+ 4	16	+ 9
7×7	24	+ 4	25	+ 9

Notice that the border-cubes pattern grows at a constant rate (4 squares each time), while the inner-cubes pattern does not. The inner-cubes pattern starts with 1 cube, then 3 more are added, then 5 more are added, then 7 more, and so on. The inner-cubes growth is represented by odd numbers, but notice that when consecutive odd numbers starting with 1 are added together, the sum is a square number. Examples: $1 + 3 = 4$, $1 + 3 + 5 = 9$, $1 + 3 + 5 + 7 = 16$.

The diagram below offers a visual representation of the growth pattern of the inner cubes.



GETTING STARTED

(Part 1 of the Three-Part Lesson)

INTRODUCING THE PROBLEM

The Getting Started part of the lesson provides an opportunity for students to activate their prior knowledge about exploring and generating patterns. Distribute 30 cubes or colour tiles, half in one colour and half in a second colour, to each student. Ask students:

What pattern can you create with coloured tiles? When you have finished, turn to your learning partner and ask them to analyse your pattern to discover the rule for your pattern.

Some anticipated student responses might include:



By observing the students' patterns and the questions they ask each other to determine the rule, the teacher assesses the range of thinking about patterns. The teacher can think about which students are using colour only as their pattern, which students show growth patterns in their work, and which students selected repeating patterns in their work. Creating the patterns activates the students' knowledge about patterns before engaging in the focus problem of the lesson, and it provides the teacher an opportunity to think about how the lesson will engage each learner in further development of their understanding.

WORKING ON IT

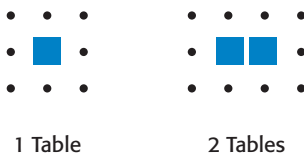
(Part 2 of the Three-Part Lesson)

Organize students for whole-group discussion (e.g., sitting in a circle at their desks or on the floor). Pose this problem for the students:

The Grade 1 teacher has asked our class to help her with a problem. She is planning a picnic for her class, and each student is going to bring one visitor. She started planning the seating by drawing a picture of the picnic table. (Draw Table 1 on the board.)

She thinks that 8 people can sit at one table, but that won't be enough seating for every student and every guest. She wants everyone to be able to sit at the same table.

If she pushed two tables together, 10 people could sit down.



How many picnic tables will the Grade 1 class need to seat 30 people?

STAGE ONE: UNDERSTAND THE PROBLEM

Whole Group: Ask the students to turn to a partner and describe what they think the problem is asking them to do? Ask for a volunteer who feels able to explain to the class what the problem is asking. After the student provides the explanation, ask the class, "Is there anyone else who can explain it in a different way?" This exchange allows students to think about and articulate the problem. Tell students, "If you understand the problem, show me a 'thumbs up'; if you do not understand the problem, show me a 'thumbs down'." The teacher is able to quickly assess who understands the problem and is ready to work. Continue the conversation until all the students fully understand the problem.

STAGE TWO: MAKE A PLAN

Small Group: Provide each small group (3 or 4 students) with centimetre grid paper, markers, interlocking cubes, and colour tiles. Remind students that they may use any other tools or materials (manipulatives, calculators, or found materials) they want in order to solve the problem.

Allow students time to organize themselves and to discuss their thinking about how their group will solve the problem.

STAGE 3: CARRY OUT THE PLAN – SOLVE THE PROBLEM

Small Group: Students continue to work on solving the problem in their small groups. As students work, the teacher observes the groups and asks them questions to help them focus their thinking. It is critical that the teacher refrain from telling students how to solve the problem or from providing hints, as this will interfere with the students' construction of their own understanding. The teacher may ask questions such as:

- Can you tell me what you are doing?
- I don't understand that; can you explain it to me again?
- What are you noticing about the number of tables and the number of guests?
- Why did you choose to use the table of values to work on the problem?
- I'm still not sure if I understand how many picnic tables you think you will need. Can you tell me again?

STAGE FOUR: LOOK BACK AND REFLECT

Small Group: Ask each group of students to hang their solution on the board or wall. Provide students with sticky notes and direct them to take a gallery walk around the room to view each solution. While they are viewing the solutions, ask them to put any questions they have about the solutions on sticky notes and attach them to the group's solution.

Whole Group: Provide opportunities for the students to meet with their groups to discuss the questions on the sticky notes. Ask each group to answer the sticky note questions in front of the whole group.

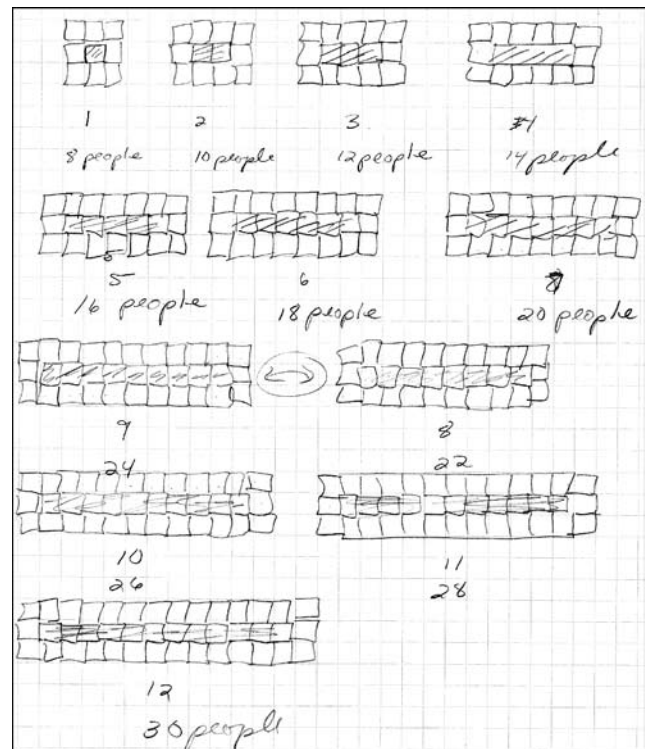
REFLECTING AND CONNECTING

(Part 3 of the Three-Part Lesson)

At this point in the lesson, students have had ample opportunity to solve the problem and record their thinking and representations on chart paper. The teacher selects three or four samples to use for the consolidation of the lesson. Different representations of the problem and the solution should be chosen to provide opportunities for rich discussion and dialogue about the mathematics. Anticipated student responses may include similar examples to these:

picnic tables	people
1	8
2	10
3	12
4	14
5	16
6	18
7	20
8	22
9	24
10	26
11	28
12	30

It goes up 2 each time.
She needs 12 tables.

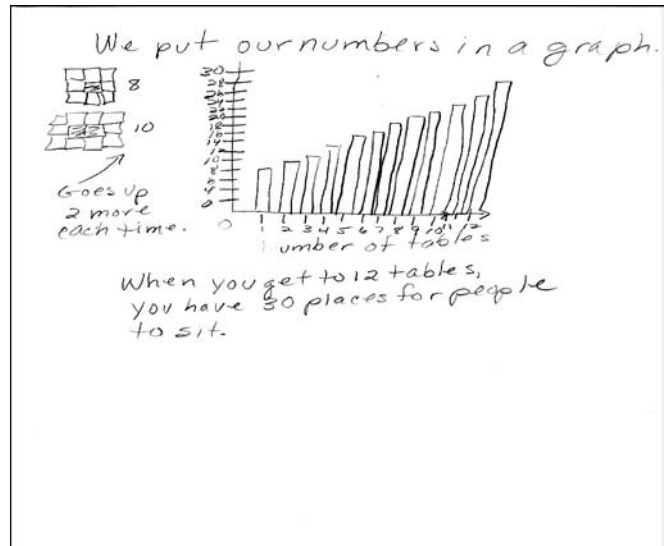


picnic tables	people
1	8
2	10
3	12
4	14
↓	↓
12	30

Grade 1's need 12 tables.



We saw that each table had six on both sides so every time we had to add 6. The double of the number of tables plus six tells how many people



Provide time for each group of students to describe their solution. Ask them how they decided to represent their thinking. As students talk about their thinking, place strips of paper over the samples to notate their work (e.g., Table of Values, Graphics, Graphs, Number Sentence). Allow the students in the class to ask the group any clarifying questions after they have explained their solution. Then ask a student from the class to repeat in his or her own words what the group explained about their solution.

To demonstrate their understanding, students may have elected to:

- construct the table shapes with manipulatives;
- draw/shade a grid;
- complete a T-chart;
- explain a pattern rule;
- extend the pattern to 12 tables.

Scaffolding questions:

Does your model match the diagram? How many pieces did you need for each part? How many more did you use each time? Was it always the same?

Students may say that they were able to find the solution by counting desks and adding tables, by using a T-chart, by seeing a pattern. Students may say that the number of tables is doubled and then 6 is added to find out how many people can sit at the tables. As students describe this, the teacher can annotate on the board:

Student says: Double the number of tables and add 6.

Teacher writes: # of tables + # of tables + 6 = number of people.

The teacher may ask students if they can think of another way to write the rule of the pattern from the problem. Student responses might include:

$2 \times$ the # of tables plus 6.

$2 \times$ tables + 6 = # of people for the picnic.

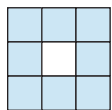
The process of using three or four samples from the students' work to discuss solutions, strategies, and representations is referred to as a "math congress". This term was coined by Cathy Fosnot in her work with elementary students. Mathematicians meet in a congress to discuss their ideas, provide proofs, and dialogue about math. The "math congress" provides students with an opportunity to learn through problem solving and creates differentiated instruction for all students in the learning environment. The congress helps students to construct the meaning of the mathematics as opposed to rigidly following rules that they may not understand.

TIERED INSTRUCTION

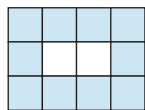
Supports and extensions can be beneficial for all students. For any given activity, there will always be some students who require more or less support, or for whom extensions will increase interest and deepen understanding.

EXTENSION

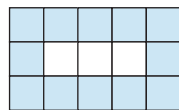
Students may want to consider the relationship discussed in the introduction to this activity. Distribute **PA.BLM4a.1**. Students will be able to build on their experience with the middle row below to try and discover the rule for the first and third rows below.



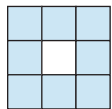
Phase 1



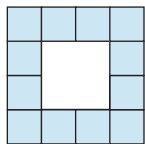
Phase 2



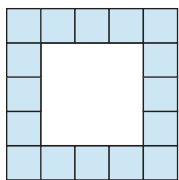
Phase 3



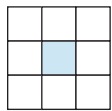
Phase 1



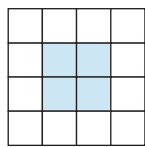
Phase 2



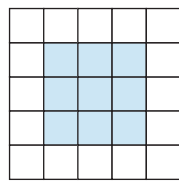
Phase 3



Phase 1



Phase 2



Phase 3

Pattern: Row 1
(as in the problem solved by students)

Phase	Pieces
1	8
2	10
3	12
4	14
5	16

Multiply the phase number by 2, then add 6 to get the number of pieces.

Pattern: Row 2

Phase	Pieces
1	8
2	12
3	16
4	20
5	24

Multiply the phase number by 4, then add 4 to get the number of pieces.

HOME CONNECTION

Extend the activity further by providing students with a different but similar pattern to work on at home. Distribute **PA.BLM4a.2** to students. This activity follows a format similar to the activity completed in class. Ask students to share with their parents what they have learned, and demonstrate their understanding of the concept using numbers, pictures, and words.

ASSESSMENT

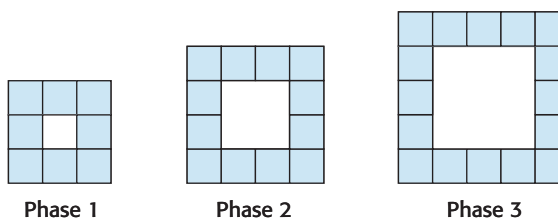
Use the students' discussions and representations to identify areas of focus for the next lesson. Assessment for learning uses observation and data collected by the teacher during the lesson to plan the next instructional focus. Assessment opportunities include:

- an interview or informal discussion during the activity;
- observation of progress as students move from one part of the lesson to the next;
- student responses to questions posed by fellow students at the math congress;
- written representations of the math on the solution charts.

SPACE STATION CHALLENGE

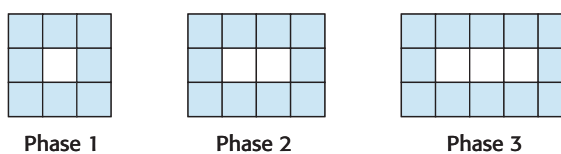
Part 1: Design A

The Canadian government wants to build an expandable space station that could provide classrooms for students who are studying outer space. Here is the design proposed by one company. The company provided a model made of cubes to show how the station might grow. The diagram shows the first three phases of the project. If the space station were to grow 2 more times, what would it look like?



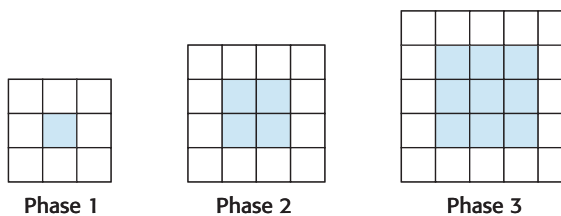
Part 2: Design B (Do not begin until Part 1 of the Project Status form is signed.)

Another company has submitted a different proposal for the space station. The first phase is also made up of 8 cubes. The model and its growth in the first three phases are represented at right. Compare Design B with Design A. At Phase 8, which design will involve more cubes? How do you know?



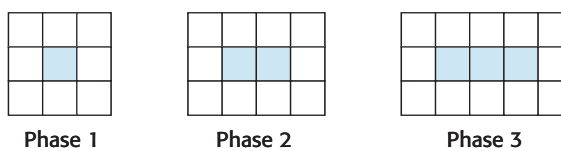
Part 3: Designs C and D

Each of the two companies also submitted a second design that is a variation on their first design. Notice that in Designs C and D the space station is constructed in the central space of Design A or Design B. Look at the central space created by Design A.



Phase 1 is the size of 1 cube, Phase 2 is the size of 4 cubes, and Phase 3 is the size of 9 cubes. How large would the central space be for phase 5? Model the problem, record your results on a T-chart, and develop a pattern rule.

Design D uses the central space of Design B. Phase 1 is the size of 1 cube, Phase 2 is the size of 2 cubes, and Phase 3 is the size of 3 cubes. How large would the central space be for Phase 5? Model the problem, record your results on a T-chart, and develop a pattern rule.



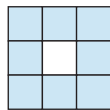
SPACE STATION CHALLENGE HOME CONNECTION

Dear Parent/Guardian:

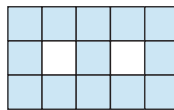
In math we are currently exploring many types of patterns. The class has already examined four possible designs for an expandable space station to provide classrooms for students who might one day study in outer space.

This home task builds on the activity by presenting a new design for consideration. Please ask your child to investigate the following problem:

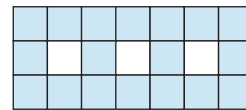
The Canadian government has just considered four proposals for the design of an expandable space station. A fifth design has been submitted, and as with the other designs, a model made of cubes demonstrates how this design would grow. The diagrams below show the first three phases of construction. If the space station were to grow 2 more times, what would it look like? How many cubes would there be in that phase (Phase 5)? What is the pattern rule?



Phase 1



Phase 2



Phase 3

Your child could solve this problem by building a model, drawing a diagram, or using a T-chart (shown on the right).

Back in class, students will be asked to share their solutions with their classmates.

Phase	Pieces
1	8
2	
3	
4	
5	

Grade 4 Learning Activity

What's My Rule?

OVERVIEW

This learning activity provides an introduction to the use of variables and examines the inverse relationship between multiplication and division. Using a literature connection, students discover the relationship between input and output values, then plot points on a graph.

Prior to this learning activity, students should have had some experience with creating charts, making graphs, modelling patterns with concrete materials, and drawing diagrams on grid paper.

BIG IDEA

Variables, expressions, and equations

CURRICULUM EXPECTATIONS

This learning activity addresses the following **specific expectations**.

Students will:

- determine, through investigation, the inverse relationship between multiplication and division (e.g., since $4 \times 5 = 20$, then $20 \div 5 = 4$; since $35 \div 5 = 7$, then $7 \times 5 = 35$);
- determine the missing number in equations involving multiplication of one- and two-digit numbers, using a variety of tools and strategies (e.g., modelling with concrete materials, using guess-and-check with and without the aid of a calculator);
- identify, through investigation (e.g., by using sets of objects in arrays, by drawing area models), and use the commutative property of multiplication to facilitate computation with whole numbers (e.g., "I know that $15 \times 7 \times 2$ equals $15 \times 2 \times 7$. This is easier to multiply in my head because I get $30 \times 7 = 210$.").

These expectations contribute to the development of the following **overall expectation**.

Students will:

- demonstrate an understanding of equality between pairs of expressions, using addition, subtraction, and multiplication.

ABOUT THE LEARNING ACTIVITY

MATERIALS

- interlocking cubes or coloured tiles
- graph paper
- chart paper
- **PA.BLM4b.1: What's My Rule? Home Connection** (1 per student)

MATH LANGUAGE



- array
- T-chart
- equality
- decompose
- symbol (variable)
- output
- input

INSTRUCTIONAL GROUPING:

whole class,
pairs

ABOUT THE MATH

In algebra terminology, a **variable** is a letter or symbol that is used to represent an unknown value (as in $5x = 15$) or a varying quantity (such as the input and output values in $P = 4s$, where output is P and input is s).

The symbols x and y are often used to represent variables. However, any letter (a or b or c ...), any symbol ( or ), or any concrete material (such as a colour tile) may be used to represent a variable.

GETTING STARTED

Start by reading *Six Dinner Sid* (Inga Moore, 1991; ISBN 0-671-7319-8), a story about a clever cat who loves to eat dinner. If the book or video is not available, use the following text to explain the premise of the story:

“The story *Six Dinner Sid* is about a very clever and manipulative cat named Sid. He has convinced six people living on the same street that he belongs to them and to them alone. As a result, he is able to enjoy six dinners a day. Unfortunately, this also means that when he gets sick, he has to visit the vet six times and take six doses of medicine, which leads to a humorous predicament.”

Discuss what else Sid might have access to in all six homes (e.g., How many people will he have to pet him? How many chairs will he have to sit on?).

WORKING ON IT

PART 1: CREATING THE T-CHART

Tell the class that they are going to focus on Sid's dinners, and use this story to explore the pattern in the number of dinners that Sid consumes, and then plot the pattern on a bar graph. Set up a T-chart on the board showing days as the input and dinners as the output. Pose the question: "How many dinners would Sid eat in one day?" Write "1" in the input column and "6" in the output column. Say, "In one day Sid would eat 6 dinners."

Input	Output
1	6
2	12
3	18
4	24
...	...
10	60

Continue prompting the class with:

- How many dinners would Sid have in 2 days? Explain.
- How many dinners would Sid have in 3 days? Can you see a pattern?
- How many dinners would Sid have in 4 days?

Typical student responses might include statements such as:

- "Every day he gets 6 more."
- "I added 6 more each day."
- "I see that you can multiply the days by 6 because he gets 6 more each day."
- "There is one more group of 6 each day."
- "It's like counting by 6s."

Prompt the students to predict the output for 5 days, and then for a full week. You might then ask, "If Sid had 60 dinners, how many days would that have taken? How do you know?"

Responses might include:

"I put 60 cubes into 6 piles and there were 10 in each pile."

"I put one cube down for dinner at each house and when I used all 60 cubes I had 10 under each house."

"I continued the pattern of adding 6. I knew he ate 42 dinners in a week, so I added 3 groups of 6 to get 60. That meant 10 days."

"I divided 60 into 6 groups, and 60 meals divided by 6 per day gives 10 days."

Teacher note: Notice the symbols used for input and output. Let the class choose the symbols for the T-chart.

Teacher note: As the students respond, prompt them to say the same thing in different ways. Remind them that multiplication is the same as repeated addition.

Teacher note: Encourage students to see that multiplying by 6 and dividing by 6 are related. If they know the input value they can multiply by 6 to determine the outcome, and if they know the output value they can divide by 6 to determine the input.

DEVELOPING A RULE

Ask for suggestions on how to reword the rule to use terms such as input value and output value.

Teacher Note: Have manipulatives available so that students can model the problem.

The chart should help students to see that the rule could be expressed in different ways:

- input value $\times 6 =$ output value
- input value + input value + input value + input value + input value + input value = output value
- output value $\div 6 =$ input value

Look at the first rule: input value $\times 6 =$ output value. Ask the class: "How could we represent the input value without having to write out the words each time? Is there a short form?" Brainstorm different ideas. Also, offer the suggestion that they could use the sun symbol (or the input symbol the class has chosen) to represent the number of days.

Now ask the class: "How could we represent the output value without having to write out the words each time? Is there a short cut?" Brainstorm ideas. One option is to use the dinner plate (or the output symbol the class has chosen).

The first rule (the input value multiplied by 6 = output value) could be written as:

$$6 \times \text{☀} = \text{🍽} \text{ or } \text{☀} \times 6 = \text{🍽}$$

CHECKING THE RULE

Tell the class, "Let's see if our rule works. What number do I put in for the sun? Let's start with 1. 1×6 is 6 and Sid does have 6 dinners in 1 day, so it works. Let's see if it works for 2 days. I put 2 in for the sun and 2×6 is 12. Yes, he has 12 dinners in 2 days. Work in your groups and see if the rule works for the whole week." Circulate and look at how the students are substituting numbers for symbols. Ask questions to see if everyone has a clear understanding.

WRITING THE RULE A DIFFERENT WAY

Say: "Let's look at another way of writing the rule. Could we write the rule without using multiplication?" The rule could be written as:

$$\text{☀} + \text{☀} + \text{☀} + \text{☀} + \text{☀} + \text{☀} = \text{🍽}$$

Teacher prompts: The sun represents the input value, the number of days. What do we have to do to the input value to get the output value? We could write $6 \times \text{☀}$ or $\text{☀} \times 6$ to show this part of the rule. What does the dinner plate represent?

Have students work in their groups to substitute values and see if the rule works for the week. Continue: "Let's look at yet another way of writing the rule. Could we write the rule without using multiplication or addition?" The final rule (output value \div 6 = input value) could be written as:

$$\text{🍴🍽️🍴} \div 6 = \text{☀️}$$

Have students work in their groups to substitute values and see if the rule works for the whole week.

Teacher prompts: "If we start with the output, which is the number of dinners, and if we divide the output into 6 groups, it equals the input. If we put in the symbol for output, the dinner plate, and we divide the output into 6 groups, we should get the input, which is the sun."

PART 2: CREATING THE SECOND T-CHART

Tell the story of Four-Meal Fred: "Fred is a St. Bernard dog with a voracious appetite. He is not quite as clever as Sid but he is able to convince four different families that he belongs to them. They all feed him, of course, so Fred gets four meals a day. You will now work in pairs to determine how many meals Fred would receive in a week."

Make clear the scope of the task. Each pair needs to:

- create a T-chart with values;
- write at least one rule in words;
- represent the rule in symbols;
- check that the rule is correct.

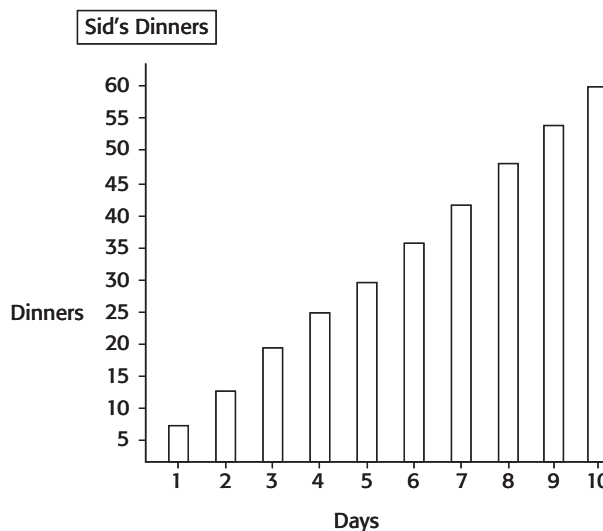
Extend the problem:

- How many meals might Fred receive in a month?
- How many days would it take for Fred to have 56 meals?

Input	Output
☐	▲
1	4
2	8
3	12
4	16
5	20
6	24
7	28

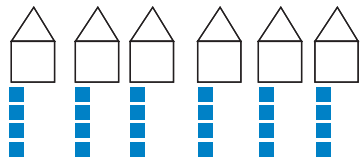
PART 3: GRAPHING SID'S DINNERS AND FRED'S MEALS

Explain to students that they will be using the data they gathered on Sid's dinners and representing it as a bar graph, with the input numbers on the horizontal axis and the output numbers shown as bars.



Have students use the data from Fred's meals to create their own graph, using graph paper or graphing software. Have them create a variety of representations for Fred's meals. They could use the following template:

Diagram or manipulative



T-Chart

Input □	Output ▲
1	4
2	8
3	12
4	16

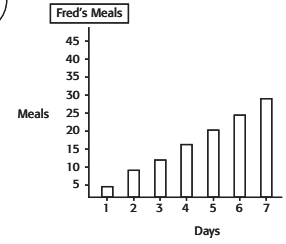
Fred's Meals

Rule

Multiply the input by 4 to get the output

$4 \times \square = \blacktriangle$

Graph



REFLECTING AND CONNECTING

Draw the input/output chart for Fred's meals on the board or on an overhead transparency, and encourage students to fill it in. Prompt the class to give a rule for what's happening. Students will have to make up their own symbols (variables). For example:

$$\square \times 4 = \blacktriangle, \quad \square + \square + \square + \square = \blacktriangle, \quad \blacktriangle \div 4 = \square, \quad 4 \times \square = \blacktriangle$$

Teacher note: Students may think that $\square \times 4 = \blacktriangle$ and $4 \times \square = \blacktriangle$ are different. Use simple examples to show the *commutative property* of multiplication (e.g., 3×4 is equal to 4×3).

TIERED INSTRUCTION

Supports and extensions can be beneficial for all students. For any given activity, there will always be some students who require more or less support, or for whom extensions will increase interest and deepen understanding.

SUPPORTS FOR STUDENT LEARNING

Scaffolding suggestions:

- For students who require a lower-level entry, provide a similar but simpler activity from the book *Two of Everything* (by Lily Toy Hong, 1993; ISBN 0-807-58157-7) which uses a doubling pattern.
- Have students draw houses on chart paper and model houses with tiles.
- Leave examples of the Sid problem posted as a model.
- Do a “gallery walk” to see what other students are doing.
- Provide a checklist, with the following steps to be completed:
 - create a T-chart with headings;
 - use manipulatives to model;
 - check for a pattern;
 - pair/share: orally describe the pattern;
 - write the rule;
 - use symbols to state the rule;
 - check by substituting a number.

EXTENSIONS

- Have students write a different problem based on Sid’s experiences.
- Have students read *Anno’s Mysterious Multiplying Jar* (by Mitsumasa Anno, 1999; ISBN 0-698-11753-0) to find the rule for the jar.
- Have students read *Multiplying Menace* (by Pam Calvert, 2006; ISBN 1-570-91890-2) and look for the secret of the stick.

Other literature connections:

- *Two of Everything*, by Albert Whitman. ISBN 0-807-58157-7
- *100 Angry Ants*, by Elinor J. Pinczes. ISBN 0-395-97123-3
- *Spaghetti and Meatballs for All*, by Marilyn Burns, 1997. ISBN 0-590-94459-2
- *The Number Devil (The Third Night)*, by Hans Magnus Enzensberger, 1997. ISBN 0-8050-6299-8

HOME CONNECTION

See **PA.BLM4b.1: What’s My Rule? Home Connection.**

ASSESSMENT

Observe students in order to assess their use of

- an efficient strategy;
- math language to explain the solution;
- appropriate symbols (variables) in pattern rules;
- appropriate diagrams/models.

WHAT'S MY RULE? HOME CONNECTION

Dear Parent/Guardian:



In math, we have recently explored a pattern based on multiples of six (6, 12, 18, 24, etc.), introduced in a book called *Six Dinner Sid*. Sid, the cat, is very manipulative and very clever.

He has convinced six people living on the same street that he belongs to them and to them alone. As a result, he is able to enjoy six dinners a day. Students recorded terms from this pattern on a chart and wrote rules to model the relationship, such as $6 \times \text{☀} = \text{||} \text{⊙} \text{||}$. Ask your child to explain the chart and the rule.

Your child's new task is to create a new rule. For example, the rule $3 \times \text{☀} = \text{||} \text{⊙} \text{||}$ might represent the number of meals your child eats each day (breakfast, lunch, and dinner).

Or $10 \times \text{☀} = \text{🛏}$ might represent the number of hours your child sleeps each day. Encourage your child to develop his or her own rule. Ask your child to complete a table such as the one on the right, leaving three of the output spaces blank.

In class, your child will be asking other students to try to complete the table and to work out the rule. To prepare for that activity, your child should try to determine the rule and describe it to you (using words, diagrams, physical materials, or mathematical symbols).

 Input	 Output
1	6
2	12
3	18
4	24
...	...
10	60

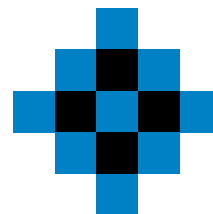
Input	Output
1	
2	
3	
4	
5	
...	...
10	

Grade 5 Learning Activity

Growing Weave Designs

OVERVIEW

In this activity, students investigate growing patterns by modelling the growth of a design, using colour tiles or interlocking cubes. They represent growth with concrete materials and they generate patterns (such as 4, 8, 12, 16, ... and 1, 4, 9, 16, 25, ...). They record patterns and generalize pattern rules using words to describe relationships. They represent the patterns in different ways, using tables, diagrams, graphs, and manipulatives. The context of the activity uses a Grade 5 art project in which students have woven different coloured papers to create a design. The teacher wonders if the students are able to recognize and represent the math found within their weave designs. Prior to this learning activity, students should have had some experience with representing patterns using charts, diagrams, graphs, and concrete materials.



BIG IDEA

Patterns and relationships

CURRICULUM EXPECTATIONS

This learning activity addresses the following **specific expectations**.

Students will:

- create, identify, and extend numeric and geometric patterns, using a variety of tools (e.g., concrete materials, paper and pencil, calculators, spreadsheets);
- build a model to represent a number pattern presented in a table of values that shows the term number and the term;
- make a table of values for a pattern that is generated by adding or subtracting a number (i.e., a constant) to get the next term, or by multiplying or dividing by a constant to get the next term, given either the list of terms (e.g., 12, 17, 22, 27, 32, ...) or the pattern rule in words (e.g., start with 12 and add 5 to each term to get the next term);
- make predictions related to growing and shrinking geometric and numeric patterns.

These expectations contribute to the development of the following **overall expectation**.

Students will:

- determine, through investigation using a table of values, relationships in growing and shrinking patterns, and investigate repeating patterns involving translations.

TIME:
2 hours

ABOUT THE LEARNING ACTIVITY

MATERIALS

- PA.BLM5a.1: The Weave Design Growth Pattern (1 per pair)
- PA.BLM5a.2: A Variation on the Weave Design Growth Pattern (1 per pair)
- PA.BLM5a.3: Graphing Design Growth Patterns (1 per pair)
- PA.BLM5a.4: Home Connection: Square Number Investigation (1 per student)
- 100 coloured tiles or interlocking cubes (50 of each colour) for each pair
- graph paper
- coloured pencils or markers

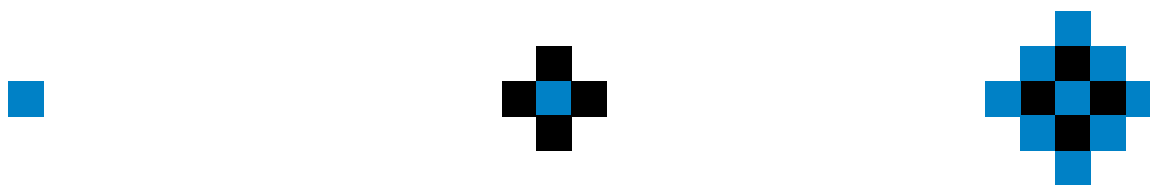
MATH LANGUAGE

- growth pattern
- array
- T-chart
- square numbers
- diagonal

INSTRUCTIONAL
GROUPING:
pairs

ABOUT THE MATH

THE DESIGN GROWTH PATTERN



We can show the growth pattern by using tiles of two different colours. In the first image, a blue tile represents the design at its starting point. In the second image, black tiles show the growth of the design that has resulted. In the third image, the black tiles are now completely surrounded by blue tiles, representing the next growth stage. The pattern alternates, with each colour surrounding the shape in turn. When modelling the pattern with tiles, you do not need to start over again to show each growth stage – just continue building onto the previous stage. The weave design grows in a number of interesting ways:

- The pattern generated by the number of Colour 1 tiles is 1, 1, 9, 9, 25, 25, This is a repeating and growing pattern that can be made by squaring the odd numbers and repeating each term twice.
- The Colour 2 tile pattern is represented by the squares of even numbers.
- The additional tiles needed to complete each stage are represented by the multiples of 4 (4, 8, 12, 16, ...).
- Counting tiles in the rows (or columns) gives the pattern $1 + 3 + 1$ for stage 2, $1 + 3 + 5 + 3 + 1$ for stage 3, and so on.
- The rule for successive stages is: stage number squared plus (stage number minus 1) squared will give the total number of tiles.

Stage	Colour 1	Colour 2	Total
1	$1 \times 1 = 1$	0	1
2	$1 \times 1 = 1$	$2 \times 2 = 4$	$1 + 4 = 5$
3	$3 \times 3 = 9$	$2 \times 2 = 4$	$5 + 8 = 13$
4	$3 \times 3 = 9$	$4 \times 4 = 16$	$13 + 12 = 25$
5	$5 \times 5 = 25$	$4 \times 4 = 16$	$25 + 16 = 41$
6	$5 \times 5 = 25$	$6 \times 6 = 36$	$41 + 20 = 61$
7	$7 \times 7 = 49$	$6 \times 6 = 36$	$61 + 24 = 85$
8
9
10	$9 \times 9 = 81$	$10 \times 10 = 100$	$145 + 36 = 181$

SQUARE NUMBERS

Square numbers are formed by multiplying a number by itself. The first seven square numbers are examined in this activity: 1, 4, 9, 16, 25, 36, and 49.

* * * * * * * * * *

* * * * * * * * *

* * * * * * *

* * * *

Square numbers get their name from the

fact that they form a square when put in an array. The square numbers can be expressed using multiplication or exponents: $1 = 1 \times 1 = 1^2$, $4 = 2 \times 2 = 2^2$, $9 = 3 \times 3 = 3^2$, $16 = 4 \times 4 = 4^2$. The meaning of such exponents can be linked to area measurement units, such as cm^2 (square centimetres) and m^2 (square metres).

Although the math above stretches beyond the Grade 5 level, understanding the richness of the patterns in this design provides the teacher with deep knowledge of the possibilities inherent in the design. Seeing the possibilities in the design enriches the teacher's ability to recognize important math learning in the students' work and dialogue which will become the underpinnings of future math study.

GETTING STARTED

(Part 1 of the Three-Part Lesson)

WARM UP: EQUAL SIGNS

Students often have a fragile understanding of the meaning of the **equal sign**. Begin the lesson with a series of equations. Write each equation one at a time on the board and ask the students to decide what the missing number is in the box. The box activity offers an opportunity to discuss variables as a representation of unknown quantities.

$$5 + 7 = \square$$

$$5 + \square = 12$$

$$\square + 7 = 12$$

$$\square + \square = 12$$

$$5 + \square = \square$$

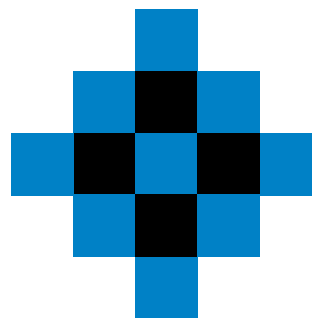
As students work through the solutions, ask them to represent or prove their answers using concrete materials. Many students misunderstand that the equal sign is a symbol between two equal values and, instead, think it means the answer is next (sum, product, quotient), regardless of where the variable occurs. On the last two equations list possible solutions that the students provide and ask students to look for patterns in the results:

$\square + \square = 12$	$5 + \square = \square$
1 11	1 6
2 10	2 7
3 9	3 8
4 8	4 9
5 9	5 10

Students benefit from ongoing opportunities to explore equations with different operations using variables in different parts of the equation.

THE GROWING DESIGN

Distribute **PA.BLM5a.1** and have students work with their elbow partner to build the first three stages of the design as shown on **PA.BLM5a.1** (see also on the right). Ask them to record the information on the chart. Circulate among the students and ensure that they are completely surrounding the design on each stage and are correctly recording the information. As the pairs of students build each stage, ask:



- How might you describe the pattern to someone who can't see it?
- How many tiles of each colour are you adding?
- Predict how many tiles you'll need for stage 4.
- Did you notice how much the total number of tiles increases from stage to stage?
- Do you think you have enough tiles to complete stage 7?

Teacher Note: Constructing the design using manipulatives is critical for several reasons. Students link patterns to spatial thinking and geometry. Visual representations help to develop abstract thinking. Furthermore, students require a concrete representation in order to make sense of the pattern.

WORKING ON IT

(Part 2 of the Three-Part Lesson)

EXPLORING GROWTH PATTERNS

Have students continue to work in pairs to complete stages 4 through 7, recording how many tiles of each colour they used and totalling the number of tiles used at each stage. Prompt students to discuss what they are doing and what patterns they see as they build successive stages (see the About the Math section). They will need to use the patterns they discover in order to extend the pattern to stage 10. Ask:

- “What is the pattern in the number of additional tiles needed to create each stage?”
- “What is the relationship between the stage number and the number of one of the colours of tiles added?”
- “How many tiles do you predict will be needed for stage 10?”

Some students may notice that the number of tiles on the side of the square is the stage number, so the number of tiles of the “outside” colour in the square is the stage number multiplied by itself: stage number \times stage number. This is a square number and can be expressed as the stage number squared.

Other students might look at the overall pattern as the number of tiles increases. Encourage these students to find the difference between the totals in order to discover the number of tiles added each time. When recording the differences they may also notice that the pattern increases by multiples of 4 (4 is added first, then 8, then 12 then 16, and so on).

Shown on the right is the design completed to the end of stage 7. There are 49 blue tiles and 36 black tiles for a total of 85 tiles. Some students may notice that there are 7 blue tiles along one edge and that this is the 7th stage, for a total of 7×7 or 49 blue tiles.

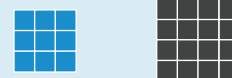
Remind students that they need to draw on graph paper a diagram of the growth of each stage, using coloured pencils or markers.

REFLECTING AND CONNECTING

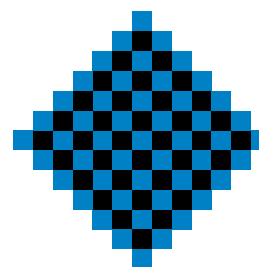
(Part 3 of the Three-Part Lesson)

When students have completed predicting the number of tiles needed for stage 10, initiate a class discussion to share different pattern rules. Ask students to check that each suggested rule actually works. Make a chart of rules that were used to solve the problem and post it in the class. See the About the Math section for a list of possible rules.

Visual Scaffolding: Have the student complete stage 4. Ask the student to decompose the tiles into two piles, according to colour, and then to arrange each colour of tiles into an array.



There should be 9 tiles of one colour and 16 of another, both arrays forming squares. Repeat for stages 5 and 6.



Teacher note: The students' suggested rules may not always be the most efficient. However, students need to "work to make sense of the problem in their own way. They (should) look for patterns and for connections with other problems" (*Teaching and Learning Mathematics: The Report of the Expert Panel on Mathematics in Grades 4 to 6 in Ontario, 2004*). After this exploration stage, you can then help students move towards more efficient methods.

TIERED INSTRUCTION

Supports and extensions can be beneficial for all students. For any given activity, there will always be some students who require more or less support, or for whom extensions will increase interest and deepen understanding.

SUPPORTS FOR STUDENT LEARNING

- Some students may need to use a calculator to help them discover patterns such as the difference or sum of different colours of tiles.
- Some students may complete the growth sequence but not identify the pattern of coloured tiles. They would benefit from deconstructing the pattern and assembling the tiles into arrays.

EXTENSIONS

- **The 20th stage.** Ask students who finish early to use their rule to find the number of tiles in the 20th stage without having to extend the chart. Solving this new problem will help to deepen their understanding of the pattern.
- **A variation on the design growth pattern.** A variation of the activity can be found in **PA.BLM5a.2**. Provide students with tiles of a third colour and ask them to add tiles in this colour to the existing weave patterns to create squares. Ask them to model the first three stages, just as they did in the original design growth activity. They should record their results in the table of **PA.BLM5a.2**, then look for patterns and develop rules for sequences. A completed table is shown on the following page. Students may find the following patterns:
 - The number of Colour 3 tiles is a multiple of 4 (0, 4, 12, 24, ...).
 - The total area is always the square of an odd number.
 - The perimeter begins at 4 and increases by 8 for each stage (4, 12, 20, 28, ...).
 - The total area is given by (number of Colour 3 tiles used) $\times 2 + 1$. For example, for stage 3, total area = (12 Colour 3 tiles) $\times 2 + 1 = 25$.



Stage	Number of Colour 3 Tiles used	Total Area	Perimeter
1	0	1	4
2	4	9	12
3	12	25	20
4	24	49	28
5	40	81	36
...
10	180	361	76

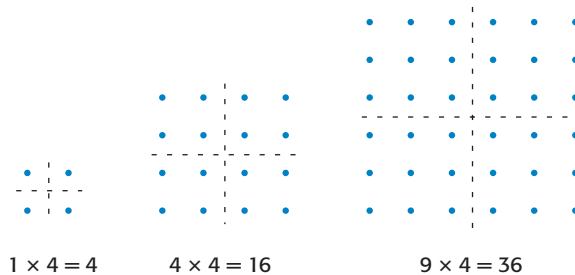
- Graphing the patterns.** For this extension, students will need a completed **PA.BLM5a.1**, **PA.BLM5a.3**, and a calculator to help them identify pattern rules represented by graphs. Ask them to choose from a variety of given graphs and to match the appropriate graphs to the patterns discovered for Colour 1, Colour 2, and the Total Area patterns (**PA.BLM5a.1**). Ask students to justify their choices. [Solutions: Colour 1 – Graph A; Colour 2 – Graph F; Total Area – Graph D.] Challenge students who finish early to work out a pattern rule for one of the incorrect graphs. Only Graph B has a pattern rule. The pattern increases as follows: 1, 3, 6, 10, 15, 21. The terms are increased by adding consecutive numbers; for example: $1 + 2 = 3$, $1 + 2 + 3 = 6$, $1 + 2 + 3 + 4 = 10$.

HOME CONNECTION

See **PA.BLM5a.4: Home Connection: Investigating Square Numbers.**

Reviewing the home connection results.

Students should have noticed that multiplying a square number by another square number will always result in a square. To reinforce this point, reproduce on the board



the diagram on the right. Explain that the rule can also be seen by looking at the factors for the numbers involved. For example, 9×4 can be written as $3 \times 3 \times 2 \times 2$. To be able to create a square number, we have to be able to regroup the factors so that they result in the product of two identical numbers, such as: $3 \times 3 \times 2 \times 2 = 3 \times 2 \times 3 \times 2 = 6 \times 6$. This type of result is never possible when multiplying a square number by a non-square number. For example, 9×6 can be expressed as $3 \times 3 \times 3 \times 2$. We cannot regroup these factors to form a product of two identical numbers.

ASSESSMENT

The completed home connection task can be used to assess the student's thinking process. A brief conversation with each student will reveal his or her level of understanding. The written paragraphs requested in **PA.BLM5a.2** and **PA.BLM5a.4** can be used to judge the student's communication skills and ability to use appropriate mathematical language.

THE DESIGN GROWTH PATTERN

An art teacher asked Grade 5 students to explore different ways to cut and weave paper together. When they showed their weaves to their homeroom teacher, the teacher noticed patterns in their work. He wondered if the students could represent the patterns in their weaves with tiles.

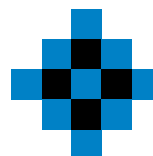
Your task is to model the growth of the design, using tiles or interlocking cubes, and then to record your results in a diagram and chart. You will need 2 different colours of tiles or cubes. The first three stages of growth are shown below.



Stage 1



Stage 2



Stage 3

Use your tiles or cubes to extend the growth pattern to stage 7, and record your results in the chart. Draw a diagram on graph paper to show the growth.

Stage	Number of Colour 1 Tiles Used	Number of Colour 2 Tiles Used	Total Number of Tiles Used
1			
2			
3			
4			
5			
6			
7			

What different patterns can you see? How can you use those patterns to help you predict:

- the number of Colour 1 tiles at stage 10?
- the number of Colour 2 tiles at stage 10?
- the total number of tiles at stage 10?

A VARIATION ON THE DESIGN GROWTH PATTERN

Here, you will use the design patterns generated in the previous activity to extend your understanding of patterns and to explore the concept of area.

Begin by adding another set of colour tiles (light blue in these diagrams) to the existing weave patterns to create squares. You may use the diagrams you created in the previous activity to help you.



After you have modelled the stages, complete the following chart. Record your results, look for patterns, and develop rules for the patterns.

Stage	Number of Colour 3 Tiles Used	Total Area	Perimeter
1			
2			
3			
4			
5			

Develop pattern rules for:

- the number of Colour 3 tiles used;
- the total area;
- the perimeter.

Are there any relationships between the columns?

Using the pattern rules you've discovered, determine the number of Colour 3 tiles, the total area, and the perimeter for stage 10. Think about how you could do this without extending the chart.

Write a brief paragraph explaining how you found the area of each of the squares at each stage.

GRAPHING THE DESIGN PATTERNS

Match the graphs (on p. 75) that represent the data from the Design Pattern activity (PA.BLM5a.1).

The graph that represents the growth pattern for Colour 1 is: _____

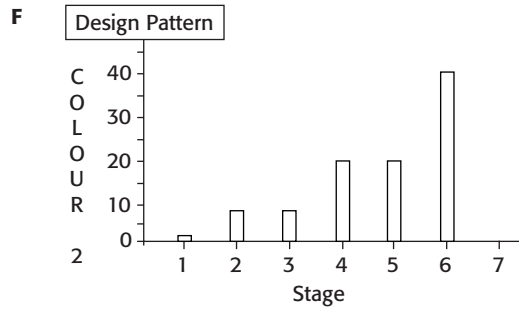
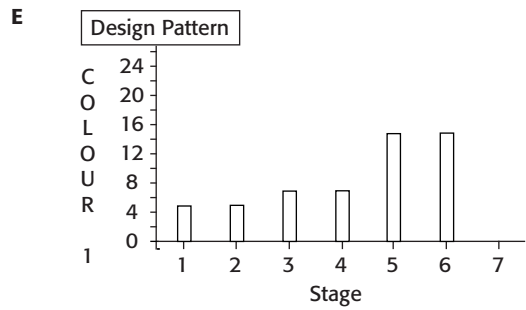
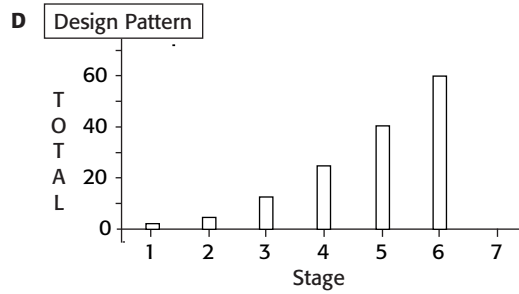
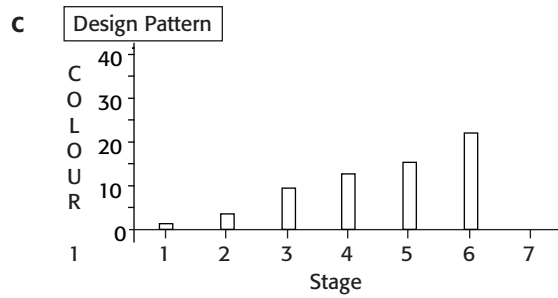
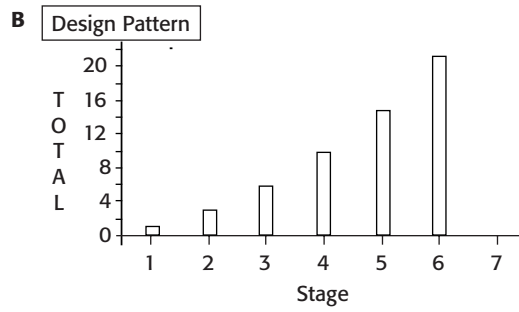
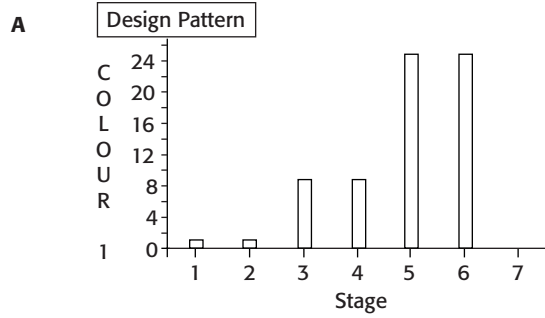
I believe that I am right because: _____

The graph that represents the growth pattern for Colour 2 is: _____

I believe that I am right because: _____

The graph that represents the growth pattern for the Total is: _____

I believe that I am right because: _____

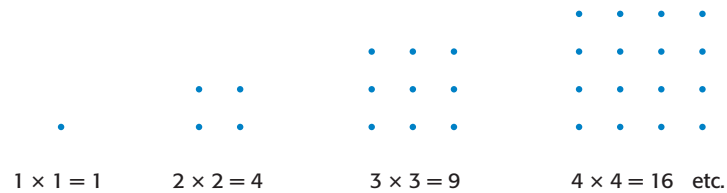


HOME CONNECTION: SQUARE NUMBER INVESTIGATION

Dear Parent/Guardian:

In math we are exploring many types of patterns. Square numbers are formed by multiplying a number by itself. They are called square numbers because they can form a square when drawn as an array of dots (see below).

The square-number pattern is:



This home connection activity builds on the work we have just completed in class. Please ask your child to demonstrate his or her understanding by writing a paragraph on each of the statements below. The paragraphs may include diagrams and/or numbers to help illustrate the reasoning behind the responses.

- If a square number is multiplied by another square number, the result is always a square number. Do you agree or disagree? Explain your thinking.
- If a square number is multiplied by a non-square number, the result is sometimes a square number. Do you agree or disagree? Explain your thinking.

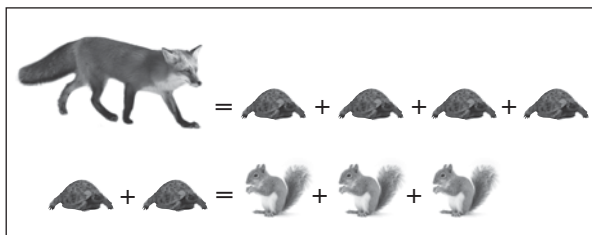
In class, students will be sharing their paragraphs with their classmates.

Grade 5 Learning Activity

Balancing Act

OVERVIEW

In this learning activity, students explore the concept of equality and variables. In the context of a literature connection, students use symbols and letters to represent the masses of different animals and to investigate relationships and express them algebraically. Students identify relationships



between the masses of the animals and write equations that represent balanced tug-of-war teams. This lesson will employ the Bansho technique of interpreting and analysing peer solutions to deepen understanding. In a Bansho, each group of students shows their work to the class. The solutions are hung on the board for all the class to see, and comparisons are made between solutions. Similar solutions are hung together, and the samples are annotated by the teacher. The teacher may choose to organize the student samples by strategy, number of solutions, chosen representations, or communication. The Bansho is used not to level the work, but to analyse the work to make math concepts explicit. Prior to this learning activity, students should have had some experience in using variables (symbols or letters) in expressions, and in modelling with concrete materials.

BIG IDEA

Variables, expressions, and equations

CURRICULUM EXPECTATIONS

This learning activity addresses the following **specific expectations**.

Students will:

- demonstrate, through investigation, an understanding of variables as changing quantities, given equations with letters or other symbols that describe relationships involving simple rates (e.g., the equations $C = 3 \times n$ and $3 \times n = C$ both represent the relationship between the total cost (C), in dollars, and the number of sandwiches purchased (n), when each sandwich costs \$3);
- demonstrate, through investigation, an understanding of variables as unknown quantities represented by a letter or other symbol (e.g., $12 = 5 + \square$ or $12 = 5 + s$ can be used to represent the following situation: "I have 12 stamps altogether and 5 of them are from Canada. How many are from other countries?");

- determine the missing number in equations involving addition, subtraction, multiplication, or division and one- or two-digit numbers, using a variety of tools and strategies (e.g., modeling with concrete materials, using guess and check with and without the aid of a calculator).

These expectations contribute to the development of the following **overall expectation**.

Students will:

- demonstrate, through investigation, an understanding of the use of variables in equations.

TIME:
2 hours

ABOUT THE LEARNING ACTIVITY

MATERIALS

- **PA.BLM5b.1: Fair Tug-of-War Teams** (1 per pair)
- **PA.BLM5b.2: Teeter-Totter Equations** (1 per pair)
- **PA.BLM5b.3: Home Connection: A Massive Puzzle** (1 per student)
- chart paper
- markers
- manipulatives such as base ten blocks, interlocking cubes, coloured tiles

MATH LANGUAGE

- equation
- variable
- identity
- infinite

INSTRUCTIONAL GROUPING:
whole group,
small group

ABOUT THE MATH

VARIABLE

A quantity, represented by a letter or symbol, that can take on one or more values. For example, in $P = 4s$ (the formula for the perimeter of a square), P and s are variables and 4 is a constant. Variables can be placeholders for missing numbers (as in $\square + \square = 14$ and $2x = 8$) or they can represent quantities that vary (as in $P = 4s$ and $y = 2x + 1$).

ALGEBRAIC EQUATION

A mathematical sentence that contains a variable or variables and an equal sign (e.g., $4 \times \square = 12$, $x + y = 15$). Solving an equation means finding values for the variable that make the equation true.

Teacher note: When working with equations, it is important to put variables on both sides of some of the equations, so that students will come to recognize that equations are a balancing across the equal sign of both numbers and variables.

GETTING STARTED

(Part 1 of the Three-Part Lesson)

WARM UP: EQUAL SIGNS

Students often have a fragile understanding of the meaning of the **equal sign**. Begin the lesson with a series of equations. Write each equation one at a time on the board and ask the students to decide what number should go in the box. The box activity offers an opportunity to discuss variables as a representation of unknown quantities.

$$\begin{aligned}5 + 7 &= \square \\5 + \square &= 12 \\ \square + 7 &= 12 \\ \square + \square &= 12 \\5 + \square &= \square\end{aligned}$$

As students work through the solutions, ask students to represent or prove their answers using concrete materials. Many students misunderstand that the equal sign is a symbol between two equal values and, instead, think it means the answer is next (sum, product, quotient), regardless of where the variable occurs. On the last two equations list possible solutions that the students provide and ask students to look for patterns in the results:

$\square + \square = 12$	$5 + \square = \square$
1 11	1 6
2 10	2 7
3 9	3 8
4 8	4 9
5 9	5 10

Students benefit from ongoing opportunities to explore equations with different operations using variables in different parts of the equation.

WORKING ON IT

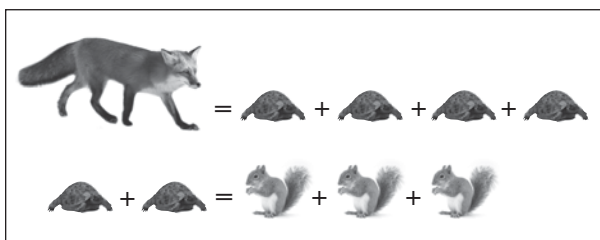
(Part 2 of the Three-Part Lesson)

Read the book *Equal Shmequal* (by Virginia Kroll, 2005; ISBN 1-570-91892-9). Pause to discuss with students the mathematics that emerges. Use letters to represent the animals in the story and, with help from the students, write equations to represent the balancing of their masses. If you do not have access to the book, read the following premise to the class:

"A little brown squirrel sits on a branch of a tree in a park, watching children play. He hears one of the children say that teams have to be equal for the tug-of-war game they are about to play. The children are called away, leaving their rope behind. The squirrel is joined by a fox, four turtles, and two more squirrels. They decide to use the rope to play a tug-of-war game. The first squirrel insists that the teams have to be equal. They decide that the teams should have equal mass. They use a teeter-totter as a balance scale to test their weights. After various trials, they manage to make the teeter-totter balance. Using the teeter-totter, they discover the following relationships between their masses:

Teeter-totter note:

Students will likely know that the balance can be thrown off if you sit closer to the centre (fulcrum) of the teeter-totter. For the animals to balance their masses accurately, they will need to be standing on top of one another.



Teacher note: Although mass is considered directly proportional to strength for the purposes of this activity, such a relationship is not always the case in real life.

“The two balanced teams then compete in a very even contest that neither side can win.”

Scaffolding suggestion:

To model the relationship, students may use manipulatives such as paper cut-outs of the animals, or the letters *F*, *T*, and *S* written on square tiles.

WRITING THE EQUATIONS

Tell students to work with their elbow partner to draw a diagram or an equation showing how the masses of the animals in the story are balanced. Ask the class to brainstorm ways they could represent each animal without having to draw each animal. Students may suggest using letters, shapes, or graphics. Students’ anticipated responses may include:

F for fox, *T* for turtle, and
S for squirrel.

OR □ = Fox ● = Turtle ▲ = Squirrel

□ = ● + ● + ● + ●

● + ● = ▲ + ▲ + ▲

$$F = T + T + T + T$$

$$T + T = S + S + S$$

Students will select symbols or variables that make sense to them.

Pose the problem for students:

How many different ways can the animals combine on either side of the teeter totter to make the teeter totter balance if you have a fox, 4 turtles, and 6 squirrels?

UNDERSTANDING THE PROBLEM

Ask the students to turn to a classmate and explain what they think the problem is asking them to do. Ask students to place their hand on the top of their head if they feel sure they understand the problem, and ask students who are still wondering to put their fist under their chin as if they are still thinking (“hmmm”). Observe the class to see if students feel they fully understand the problem. Ask a student with hand on head to explain the problem. Ask a student who was not sure to reiterate what the first student said and to ask any questions for clarification. Survey the class again; organize students into groups of 3 or 4 once the class understands the problem.

MAKE A PLAN: CREATING NEW EQUATIONS

Observe the groups as they begin their work. Ask students at each group to articulate their strategy for creating their balance equations. Use a sticky note to record the strategies students are able to articulate, and also note those strategies that students are employing but are not able to describe. If some groups have trouble beginning, ask a few groups to share the strategy they have chosen with the whole class.

CARRY OUT THE PLAN

As students begin to record their equations, ask students to explain their solutions. Resist the urge to direct students to specific equations. Deeper learning can be achieved by asking questions such as:

- How is this side of the equal sign balanced with the other side?
- I'm still not sure I understand your thinking, can you think of another way to show me?
- I see you have drawn manipulatives to show your equations; is there another way to record your thinking?
- How do you know the equation is true?





Continue to observe the students as they work, and call the group back together when you feel the groups have exhausted the solutions they are able to create.

LOOKING BACK





Reconvene the whole group and ask the groups to hang their charts on the board. Ask the groups to look at the charts and note how the solutions are the same or different. Possible student responses may include:

- "That group used the same symbols to represent the animals."
- "These two groups both made the same number of equations."
- "This group showed the same idea in a different way."

For example, if the group decided to organize by strategies, the Bansho could be organized as below:

$f=t+t+t+t$ $t+t=s+s+s$ <hr/> $t+t+t+t=f$ $t+t+s+s+s=f$ $s+s+s+s+s=f$ $f=t+t+s+s+s$ $f=s+s+s+s+s$ $f=t+t+t$ $s+s+s+s+s=f$ $s+s+s+t+t=g$	$f=t+t+t+t$ $t+t=s+s+s$ <hr/> $f=s+s+s$ $f=t+t+t$ $f=s+s+s+s+s$ $t+t$ $t+t$ $f=t+s+s+s$ $f=t+t+s+s+s$	$f=t+t+t+t$ $t+t=s+s+s$ <hr/> one Fox = 4 Turtles 2 Turtles = 3 Squirrels 3 Squirrels = 2 Turtles 4 Turtles = one Fox 6 Squirrels = 4 Turtles 6 Squirrels = one Fox	$f=t+t+t+t$ $t+t=s+s+s$ <hr/> <table border="1"> <thead> <tr> <th>F=</th> <th>T</th> <th>S</th> </tr> </thead> <tbody> <tr> <td>1</td> <td>4</td> <td></td> </tr> <tr> <td>1</td> <td>2</td> <td>3</td> </tr> <tr> <td>1</td> <td></td> <td>6</td> </tr> <tr> <td></td> <td>4</td> <td>6</td> </tr> </tbody> </table>	F=	T	S	1	4		1	2	3	1		6		4	6	$f=t+t+t+t$ $t+t=s+s+s$ <hr/> $f=$  $4t=$  $2t + 3s=$  $6s=$  so $f=t+t+s+s+s$ $f=s+s+s+s+s$	$f=t+t+t+t (1/4)$ $t+t=s+s+s (1/3)$ <hr/> $F=1/4+1/4+1/4+1/4$ $F=1/2+1/2$ $1/2+1/2=1/3+1/3+1/3$ $1/2+1/2+1/3+1/3+1/3$ $=1/4+1/4+1/4+1/4$
F=	T	S																		
1	4																			
1	2	3																		
1		6																		
	4	6																		

The teacher would listen to the groups describe their work and annotate on slips of paper the strategy the students used:

Organized List	Guess and Test	Matching Values	Make a Chart	Concrete Division	Making Fractions															
$f=t+t+t+t$ $t+t=s+s+s$ ----- $t+t+t+t=f$ $t+t+s+s+s=f$ $s+s+s+s+s=f$ $f=t+t+s+s+s$ $f=s+s+s+s+s$ $f=t+t+t$ $s+s+s+s+s=f$ $s+s+s+t+t=g$	$f=t+t+t+t$ $t+t=s+s+s$ ----- $f=s+s+s$ $f=t+t+t$ $f=s+s+s+s+s$ $t+t$ $t+t$ $f=t+t+s+s$ $f=t+t+s+s+s$ $t+t$	$f=t+t+t+t$ $t+t=s+s+s$ ----- one Fox = 4 Turtles 2 Turtles = 3 Squirrels 3 Squirrels = 2 Turtles 4 Turtles = one Fox 6 Squirrels = 4 Turtles 6 Squirrels = one Fox	$f=t+t+t+t$ $t+t=s+s+s$ ----- <table border="1"> <thead> <tr> <th>F</th> <th>T</th> <th>S</th> </tr> </thead> <tbody> <tr> <td>1</td> <td>4</td> <td></td> </tr> <tr> <td>1</td> <td>2</td> <td>3</td> </tr> <tr> <td>1</td> <td></td> <td>6</td> </tr> <tr> <td></td> <td>4</td> <td>6</td> </tr> </tbody> </table>	F	T	S	1	4		1	2	3	1		6		4	6	$f=t+t+t+t$ $t+t=s+s+s$ ----- $f=$  $4t=$  $2t + 3s=$  $6s=$  so $f=t+t+s+s+s$ $f=s+s+s+s+s$	$f=t+t+t+t$ (1/4) $t+t=s+s+s$ (1/3) ----- $F=1/4+1/4+1/4+1/4$ $F=1/2+1/2$ $1/2+1/2=1/3+1/3+1/3$ $1/2+1/2+1/3+1/3+1/3$ $=1/4+1/4+1/4+1/4$
F	T	S																		
1	4																			
1	2	3																		
1		6																		
	4	6																		

As the class discusses the Bansho, opportunities may arise to discuss other math concepts.

REARRANGING THE ANIMALS

Write the following equation on the board.

$$F + S = T + T + T + T + S$$

We know this equation is true because we are given that $F = T + T + T + T$. Adding a squirrel (S) to both sides maintains the balance. Now rearrange the letters on each side of the equation, as follows:

$$S + F = T + T + T + T + S$$

$$F + S = T + T + S + T + T$$

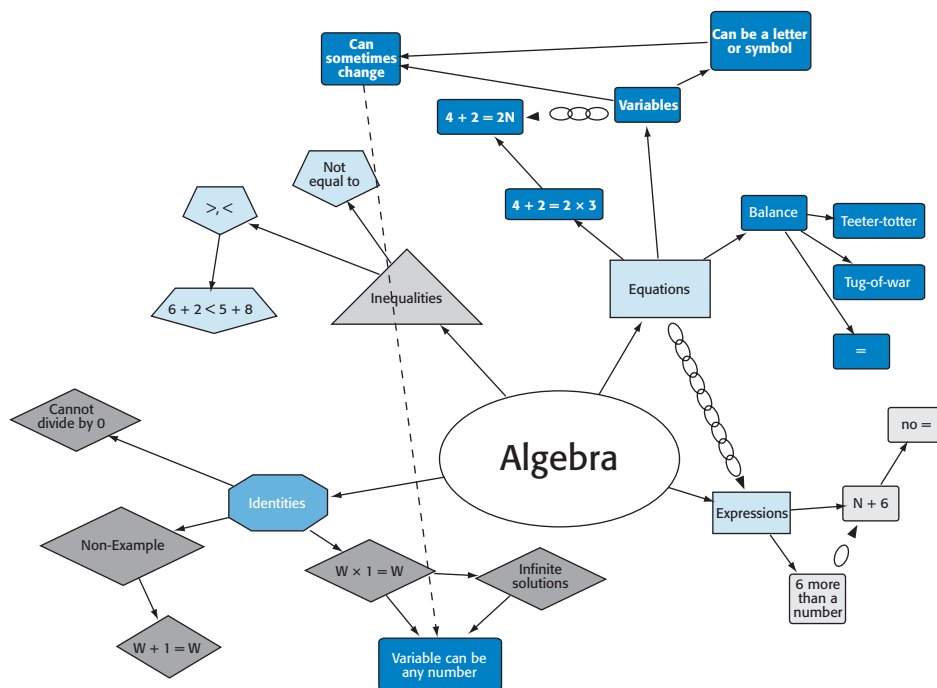
Ask students to consider whether the new equations are true. Have them work in pairs to rearrange the letters to create other equations. Share and discuss the new equations in a whole-class setting.

Rearranging the order of the masses that are added together does not change their sum. For example, $3 + 5 = 5 + 3$. This is the commutative property of addition. Does the commutative property also hold true for other operations? (It does for multiplication but not for subtraction or division.)

REFLECTING AND CONNECTING

(Part 3 of the Three-Part Lesson)

Create a word web wall with “Algebra” at the centre during the unit. Ask students to add connections after each lesson to the web. Below is a sample word web created using Smart Ideas™.



BALANCING EQUATIONS WITH NUMBERS

Write the number 14 on the board. Then ask: “What numbers could I add to get 14?” Write students’ suggestions on the board.



Their suggestions will include $1 + 13$, $2 + 12$, $3 + 11$, and so on. Draw a balance scale on the board and write $1 + 13$ on one side. Then ask, “What could I put on the other side to create a balance?” Students may want to answer 14. While the answer is correct, ask for other possibilities (e.g., $3 + 11$). Discuss how $1 + 13$ is equal to $3 + 11$. Have students brainstorm further responses. Encourage them to see that they can use more than two addends (e.g., $1 + 3 + 10$) and that they can use multiplication, division, or subtraction to create an infinite number of possibilities.

FINDING MISSING NUMBERS

Write the following equations on the board:

$$5 + u = 8 \quad 3 \times 7 = t \quad 36 \div r = 6 \quad 3 \times n + 1 = 10$$

$$2 \times t - 1 = 5$$

Ask students to work in pairs to find the unknown quantity in each equation.

Solutions:

$$5 + u = 8, u = 3$$

$$3 \times 7 = t, t = 21$$

$$36 \div r = 6, r = 6$$

$$3 \times n + 1 = 10, n = 3$$

$$2 \times t - 1 = 5, t = 3$$

Discuss the solutions and strategies with the class. Students may have used some of the following strategies:

- *Guess and test.* Example: For $3 \times n + 1 = 10$, I guessed that $n = 4$ and tried it in the math sentence. It didn't work. So I tried a value of 3 for n and put it in the sentence.
 $3 \times 3 + 1 = 10$ is correct.
- *Inverse operations.* Example: In $5 + u = 8$, we can take 5 away from both sides to get $u = 3$. This type of reasoning (adding or subtracting something from both sides of an equation) was used in the tug-of-war equations.
- *Counting up.* Example: In $5 + u = 8$, I counted 3 from 5 to 8.
- *Working backwards.* Example: In the question $2 \times t - 1 = 5$, I worked backwards. First I added 1 to 5 to get 6. Then I divided 6 by 2 and got 3. I then checked to see if I was right by substituting 3 for t and trying the math sentence. $2 \times 3 - 1 = 5$ is correct.
- *Rephrasing the question.* Example: I said 36 "divided by what" = 6.

TIERED INSTRUCTION

Supports and extensions can be beneficial for all students. For any given activity, there will always be some students who require more or less support, or for whom extensions will increase interest and deepen understanding.

SUPPORTS FOR STUDENT LEARNING

- Students will benefit from being able to use concrete materials to model the relationships explored. For example, in the tug-of-war scenarios it would be helpful to use cut-outs of pictures of the animals, or else square tiles with letters (such as F for fox), to represent each of the animals.
- The summative activity described in the Assessment section is open-ended and can help meet the needs of all children.

HOME CONNECTION

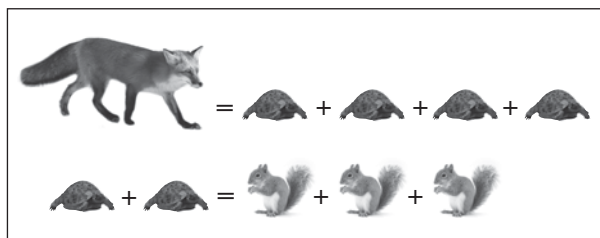
See **PA.BLM5b.3: Home Connection: A Massive Puzzle.**

ASSESSMENT

- Choose 1 or 2 equations prepared by students and ask students to explain their equation(s) in a mini interview.
- Ask students to give an example of an identity and to write an explanation of why it is an identity.
- Summative activity. Ask students to create their own tug-of-war scenario by picking the type of animals to include, deciding how many there should be of each animal, and making up relationships for their masses. Then ask students to pose two questions that may be answered based on their tug-of-war scenario.

FAIR TUG-OF-WAR TEAMS

One fox, 6 squirrels, and 4 turtles go to the park to participate in a tug-of-war game. The animals aren't sure who should participate in the tug-of-war in order for the teams to be fair. A fox has the same mass as 4 turtles, and 2 turtles have the same mass as 3 squirrels. They use a teeter totter to check if the two groups' masses are balancing equally.

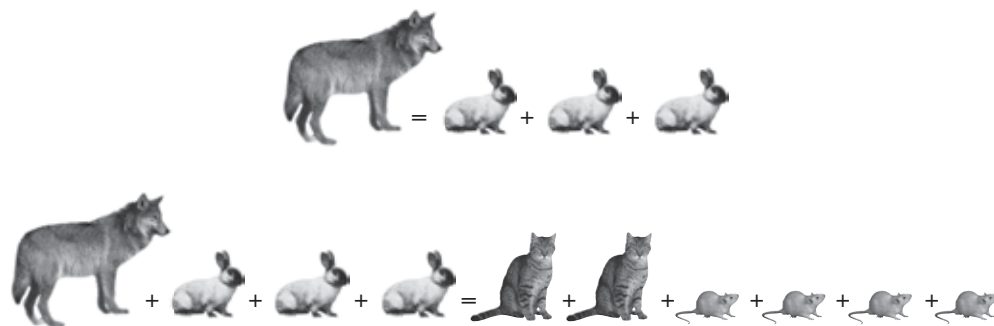


How many different ways can each side of the tug-of-war teams be organized?

TEETER-TOTTER EQUATIONS

EXTENSIONS

A similar problem introduces a new set of animals, whose mass relationships are shown below:



Students are asked: "If the wolf, being a loner, wants to play on one side of the tug-of-war rope, how many bobcats and mice will be needed on the other side to make the game fair? Explain your reasoning."

Teacher Notes:

For students to find a solution, they need to find a way of relating the two new equations:

$$W = R + R + R$$

$$W + R + R + R = B + B + M + M + M + M$$

They need to notice the following relationship:

$$W = R + R + R$$

$$W + R + R + R = B + B + M + M + M + M$$

Since 3 rabbits have the same mass as a wolf, we can regroup the right side to also be in two groups of equal masses:

$$W + R + R + R = B + M + M + B + M + M$$

This may result in students discovering that:

$$W = B + M + M$$

Many students will not use the formal reasoning described above. They will point and gesture, or might use diagrams or concrete materials, to communicate their ideas. Let the students express their thinking in their own words and in their own way but, where appropriate, model complementary strategies. For example, using equations made of letters that are coloured and circled or underlined to make relationships visually apparent is a strategy that all students should learn to use.

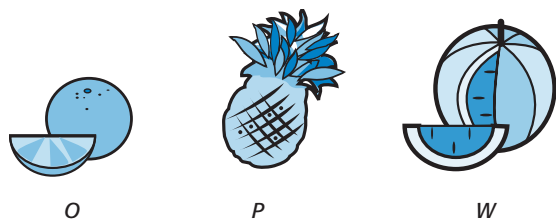
HOME CONNECTION: A MASSIVE PUZZLE

Dear Parent/Guardian,

Your child has been working on the concept of “equality” in math sentences, and on the use of variables (letters and symbols) to represent a value. The class has explored the idea of a balance scale (teeter totter) to determine whether or not the two sides of a math sentence are equal ($4 + 5 = 9$). Using strategies such as logic, “guess and test”, and “make a model”, they solved a number of balance scale problems. Encourage your child to experiment with different strategies to solve the similar problem posed below and to check through substitution of values.

FIND THE MASS OF EACH OBJECT

This problem deals with 1 bag of oranges (represented by O), 4 pineapples (P), and 1 watermelon (W). Use the clues in the table to find the mass of each object. For example, using the first row we know that the combined mass of the oranges, pineapples, and watermelon is 17 kg. Use the other rows and columns to get more clues. Write an algebraic sentence to represent each relationship (e.g., $O + P + W = 17$). Explain how you solved the problem.



			17 kg
			14 kg
			21 kg
			13 kg
19 kg	22 kg	24 kg	

Your child can practise solving similar problems at www.mathplayground.com. Click on “Weigh the Wangdoodles”.

Grade 6 Learning Activity

You're a Winner!

OVERVIEW

In this learning activity, students investigate growing patterns that result in triangular numbers (numbers such as 1, 3, 6, and 10, which can be expressed as triangular patterns of dots). They also investigate the patterns in finding the sums of triangular numbers. The activity involves the analysis of a highly unusual method of delivering a collection of miniature toys to the first-prize winner at a fun fair:

Day 1 – 1 chipmunk

Day 2 – 1 chipmunk and 2 blue jays

Day 3 – 1 chipmunk, 2 blue jays, 3 puppies

The delivery would continue each day in the same manner until the 10th day.

Day 10 – 1 chipmunk, 2 blue jays, 3 puppies, 4 kittens, 5 butterflies, 6 ducklings, 7 rabbits, 8 goldfish, 9 ladybugs, 10 caterpillars

The mathematical context is the list of triangular numbers found in Pascal's Triangle (shown on the right).

Prior to this learning activity, students should have had some experience with representing patterns and using tables, diagrams, and manipulatives.

The students will be challenged to choose to receive 25 toys immediately or to accept the unusual delivery method.

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*           *           *           *
*          * *         * *         * *
*         * * *       * * *       * * *
*        * * * *     * * * *     * * * *

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1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
...

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BIG IDEA

Patterns and relationships

CURRICULUM EXPECTATIONS

This learning activity addresses the following **specific expectations**.

Students will:

- make tables of values for growing patterns, given pattern rules in words (e.g., start with 3, then double each term and add 1 to get the next term), then list the ordered pairs (with the first coordinate representing the term number and the second coordinate representing the term) and plot the points in the first quadrant, using a variety of tools (e.g., graph paper, calculators, dynamic statistical software);
- determine the term number of a given term in a growing pattern that is represented by a pattern rule in words, a table of values, or a graph;
- describe pattern rules (in words) that generate patterns by adding or subtracting a constant, or multiplying or dividing by a constant, to get the next term (e.g., for 1, 3, 5, 7, 9, ..., the pattern rule is "start with 1 and add 2 to each term to get the next term"), then distinguish such pattern rules from pattern rules, given in words, that describe the general term by referring to the term number (e.g., for 2, 4, 6, 8, ..., the pattern rule for the general term is "double the term number");
- determine a term, given its term number, by extending growing and shrinking patterns that are generated by adding or subtracting a constant, or multiplying or dividing by a constant, to get the next term.

These expectations contribute to the development of the following **overall expectation**.

Students will:

- describe and represent relationships in growing and shrinking patterns (where the terms are whole numbers), and investigate repeating patterns involving rotations.

ABOUT THE LEARNING ACTIVITY

MATERIALS

- **PA.BLM6a.1: Anticipation Guide** (1 per group)
- **PA.BLM6a.2: Pascal's Triangle** (1 per student)
- **PA.BLM6a.3: Problems Carousel** (1 per class)
- **PA.BLM6a.4: Home Connection: Beyond Triangular Numbers** (1 per student)
- chart paper
- markers
- coloured tiles, interlocking cubes, relational rods, pattern blocks, and/or beads/buttons
- calculators

MATH LANGUAGE

- triangular numbers
- Pascal's Triangle
- sequence
- consecutive numbers
- spatial
- temporal

TIME

2 hours

INSTRUCTIONAL
GROUPING:

table groups
(approximately
4 students)

ABOUT THE MATH

HOW TO GENERATE TRIANGULAR NUMBERS

- Start with 1, add 2 to get the second number, add 3 to get the third, add 4 to get the fourth, and so on.

$$1$$

$$1 + 2 = 3$$

$$1 + 2 + 3 = 6$$

- Start with 1 dot, add 2 dots below it to form a triangle of 3 dots, then add 3 dots below that to form a triangle of 6 dots, then add 4 dots below that to form a triangle of 10 dots, and so on.



HOW TO FIND THE VALUE OF A TRIANGULAR NUMBER

If we are asked, for example, “What is the 20th triangular number?”, we can find its value without first determining the triangular numbers that precede it. From the patterns above we can see that the 20th triangular number is the sum of $1 + 2 + 3 + 4 + \dots + 18 + 19 + 20$. How can we find this sum without adding all these numbers together? One way is to notice that $1 + 20 = 21$, $2 + 19 = 21$, $3 + 18 = 21$, and so on. There are 10 such pairs, so the 20th triangular number is $10 \times 21 = 210$. Another way is to notice that a triangular number and its copy can fit together to form a rectangle.

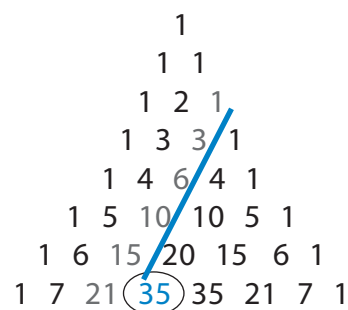


The 4th triangular number is $4 \times 5 \div 2 = 10$.

The 20th triangular number is $20 \times 21 \div 2 = 210$.

HOW TO FIND THE SUM OF THE FIRST 20 TRIANGULAR NUMBERS

The first 10 triangular numbers have 10 ones, 9 twos, 8 threes, and so on. So the sum of the first 10 triangular numbers is equal to $10 \times 1 + 9 \times 2 + 8 \times 3 + 7 \times 4 + \dots + 3 \times 8 + 2 \times 9 + 1 \times 10$. The sum is also found in Pascal’s Triangle (see description below). The diagram shows that the sum of the first five triangular numbers ($1 + 3 + 6 + 10 + 15$) is 35 (see <http://www.mathematische-basteleien.de/triangularnumber.htm>).



PASCAL’S TRIANGLE

Pascal’s Triangle is named after Blaise Pascal, the French mathematician who studied it. Interestingly, this pattern was known in China long before Pascal’s time. Notice how Pascal’s Triangle grows:

- the numbers on the outer diagonals have a value of 1;
- each of the other numbers is the sum of the two numbers directly above it (for example, the 6 in the 5th row is the sum of the two 3’s above it).

Pascal’s Triangle has many applications in mathematics and contains within itself many interesting patterns, such as triangular numbers.

GETTING STARTED

(Part 1 of the Three-Part Lesson)

ACTIVATING PRIOR KNOWLEDGE

Divide the class into groups of up to four students each. Tell students that they are going to be exploring and generating patterns. Have them complete the “Before” side of the Anticipation Guide (**PA.BLM6a.1**) as a way of activating prior knowledge and identifying common misconceptions. Then have them brainstorm applications of patterns in the real world. Patterns may be categorized as sequential (playing cards), spatial (buildings), temporal (calendars), and linguistic (spelling rules).

WORKING ON IT

(Part 2 of the Three-Part Lesson)

Introduce this scenario to the whole class:

“At a community fair, a student wins first prize – a collection of miniature toys. He has a choice to have 25 toys immediately or have toys delivered every day to his home for the next 10 days. For the next ten days the toys would be delivered to the student’s home in a most unusual way. On the first day the winner’s package contains 1 chipmunk; on the second day the package contains 2 blue jays and 1 chipmunk; on the third day the package contains 3 puppies, 2 blue jays, and 1 chipmunk; on the fourth day the package contains 4 kittens, 3 puppies, 2 blue jays, and 1 chipmunk. The same pattern continues until, on the tenth day, the package contains 10 caterpillars, 9 ladybugs, 8 goldfish, 7 rabbits, 6 ducklings, 5 butterflies, 4 kittens, 3 puppies, 2 blue jays, and 1 chipmunk. The question is: Which way of receiving the prizes is the best choice for the winner?”

UNDERSTANDING THE PROBLEM

Have students think about what the solution(s) might look like and have them rephrase the question. Ensure that manipulatives are available and invite students to use them to model the problem. Tell students that their groups will be required to display their solutions on chart paper and explain their processes to other groups and eventually to the class.

MAKE A PLAN

Circulate and ensure that each group understands the task and has a plan of action. If groups are having trouble developing strategies, allow one member to “phone a friend” (i.e., visit another group to observe other strategies). Understanding the question and developing an appropriate strategy will be challenging for some students. You might prompt students’ thinking by asking:

- What could you use in the classroom to model the problem?
- What strategies have you considered?
- How might you organize the information?

LOOKING BACK

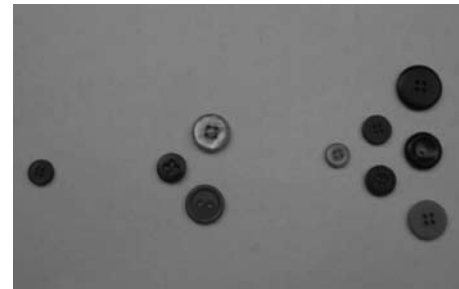
After students have solved the problem and decided what choice the winner should make, gather the students together for a whole class discussion. Each group of students should share their findings with one other group before presenting to the whole class.

Select three or four students who have different representations or strategies to explain their chart and their thinking. Some anticipated student responses might include:

Create a table of values. The problem can be clarified for some students if they create a table of values such as the one below.

Day	Toys
1	chipmunk
2	chipmunk, blue jay, blue jay,
3	chipmunk, blue jay, blue jay, puppy, puppy, puppy

Model using concrete materials. Students may choose to use beads, tiles, cubes, and so on, to build a day-by-day model on their desks. They might also use the letters A through J to represent the 10 types of toys. Thus, the toys received on the fifth day would be ABBCCDDDDDEEEEE.



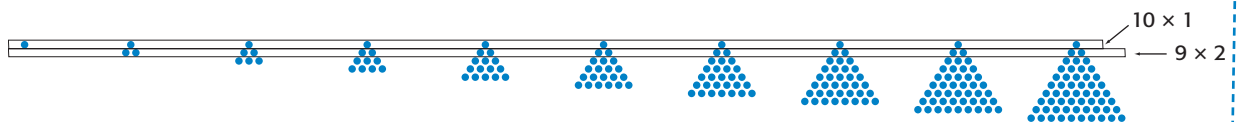
Use numerical representation. Some students might develop an algorithm, such as:

animal 1 + animal 1 + 2 × animal 2 + animal 1 + 2 × animal 2 + 3 × animal 3 + ... = 1 + (1 + 2) + (1 + 2 + 3) + ... **or**

total day 1 + total day 2 + total day 3 + ... = 1 + 3 + 6 + 10 + ... **or**

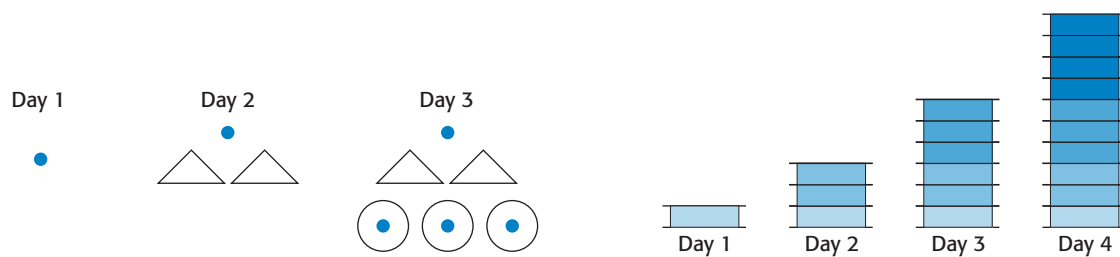
Teacher Note: Students are unlikely to use brackets in the first algorithm.

The first 10 triangular numbers have 10 ones, 9 twos, 8 threes, and so on. Therefore, the sum of the first 10 triangular numbers is equal to $10 \times 1 + 9 \times 2 + 8 \times 3 + 7 \times 4 + \dots + 3 \times 8 + 2 \times 9 + 1 \times 10$, as shown below.



Teacher note: Recognizing that $10 \times 1 + 9 \times 2 + 8 \times 3 + \dots + 2 \times 9 + 1 \times 10$ is an equivalent representation of adding together the total number of toys each day to get the sum of the first 10 days indicates a developmental leap in a student's understanding of patterning concepts.

Draw a diagram. Some students may draw symbols to represent their solution. Other students may use graph paper to draw their representation. Encourage groups to clearly explain their thought processes on chart paper. They should use numbers, pictures, and words.



REFLECTING AND CONNECTING

(Part 3 of the Three-Part Lesson)

During students' presentations, avoid implying that some strategies are better than others. Encourage students to consider the range of strategies and to try and make sense of each representation. Ask:

- How easy is your strategy to explain?
- What other strategies did you try?
- If you were to increase the number of days, would your strategy still work?
- What would you do differently if you were to solve a similar problem?

After the groups have presented their work, illustrate the pattern by saying: "I've noticed that many of your solutions are similar. Many of you recorded the number of toys per day and added up the results. Let's look at the pattern on the board."

Write the following numbers: 1, 3, 6, 10, 15, 21, 28, 36, 45, 55. Ask the students: "Which choice would give the winner the most toys?" (Solution: The winner received 120 toys in the 10 days. It was a better choice for the winner.)

Illustrate the pattern by drawing dots on the board in the shape of triangles for the first 5 numbers in the sequence. Ask the students: "What do the shapes look like?"

Explain to students that these numbers are called **triangular numbers** and were studied by mathematicians in ancient Greece and in China. Give students time to extend the pattern to the 20th day. Encourage them to brainstorm the uses of triangular numbers in real-life applications. Examples might include the set-up for 10-pin bowling, cheerleaders' pyramids, or displays of soup cans.

Ask the students to complete the second part of the anticipation guide and think about how their thinking about patterns and numbers has changed.

Correct Anticipation Guide Responses

1. Agree. Pattern increases by 4, 6, 8, 10.
2. Disagree. Pattern doubles, then increases by 1, so correct response is 15.
3. Disagree. Some do and some don't (for example: 1, 2, 4, 8, 16, ...)
4. Agree. All patterns, by definition, have rules.
5. Agree.
6. Disagree. There is no rule. (Students may argue that there is a pattern that has not repeated yet. However, no pattern is evident in the sequence given.)

TIERED INSTRUCTION

Supports and extensions can be beneficial for all students. For any given activity, there will always be some students who require more or less support, or for whom extensions will increase interest and deepen understanding.

SUPPORTS FOR STUDENT LEARNING

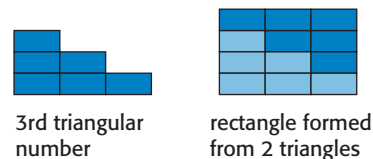
Understanding the question and developing an appropriate strategy will benefit students. You may support students by asking:

- What materials could you use in the classroom to model the problem?
- What strategies have you considered for starting to solve the problem?
- How might you organize the information you have been given?

EXTENSIONS

Some groups may complete the activity early. Challenge early finishers to determine how many toys the student would receive in 15 days if the same pattern were followed. Encourage students to develop an algebraic pattern rule.

Finding Values Of Triangular Numbers. Have students create a model of the 3rd triangular number. Have them put their model together with a partner's model to form a rectangle. Pose the following questions:



- What is the length of the rectangle? What is the width of the rectangle? What is the area of the rectangle?
- How do the length and width of the rectangle relate to the stage number? Can you use numbers or symbols to describe the pattern?
- Repeat the experiment with the 4th and 5th triangular numbers. Is there a pattern?
- How could you use the pattern to find the 20th triangular number? Any other triangular number?

Teacher note: The algebraic expression for the n th triangular number is:

$$\frac{n(n+1)}{2}$$

Finding Patterns In Pascal's Triangle. Give each student a copy of **PA.BLM6a.2**. Tell the students that they are looking at a partly completed example of Pascal's Triangle, which is a pyramid of numbers. Explain that, while the Chinese investigated properties of the triangle in the 12th century, it was Blaise Pascal, a French mathematician, who in the 16th century was the first to document this "arithmetical triangle". When presenting Pascal's Triangle to the class, start with the top two rows and ask students to consider how the third row in **PA.BLM6a.2** might be determined from the first two rows. Then ask a similar question about the fourth row. Let students arrive at their own ideas and give them opportunities to explain their reasoning. Ask: "Can you find any triangular numbers in Pascal's Triangle?"

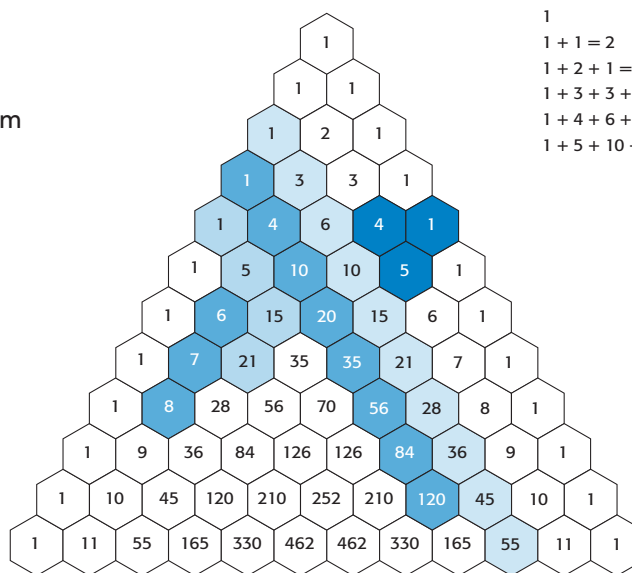
Highlight examples on the overhead transparency as students point them out. Have students highlight the same examples in yellow on their own copies. Once they have highlighted all the triangular numbers they can find, have them check the accuracy of their findings using a solution transparency.

Next, instruct students: "Find as many patterns as you can. Indicate each pattern by using a different colour. Create a legend to explain each pattern that you find."

Have students share the patterns they have found, and add the patterns, using different colours, to the image on the overhead transparency.

Some patterns that students might notice in Pascal's Triangle:

- triangular numbers – light blue line of 10
- consecutive numbers – medium blue line of 3
- two adjacent numbers having the sum of the number below – dark blue triangle
- a hockey stick pattern (e.g. $1 + 5 + 15 = 21$) – shaded light blue (the hockey stick pattern can be any length)
- horizontal sums (shown beside the main triangle)
- sums of triangular numbers (e.g., $1 + 3 = 4$, $4 + 6 = 10$) – medium blue line of 8



$$\begin{aligned}
 &1 \\
 &1 + 1 = 2 \\
 &1 + 2 + 1 = 4 \\
 &1 + 3 + 3 + 1 = 8 \\
 &1 + 4 + 6 + 4 + 1 = 16 \\
 &1 + 5 + 10 + 10 + 5 + 1 = 32
 \end{aligned}$$

Ask students to solve the problems on the activity cards in **PA.BLM6a.3: Problems Carousel**.

The solutions to the six problems are:

1. Person one shakes 6 hands, person two shakes 5 hands, person three shakes 4 hands, and so on. $6 + 5 + 4 + 3 + 2 + 1$ is 21. There are 21 handshakes. The 6th triangular number is 21.
2. The 12th triangular number is 78.
3. The 10th triangular number is 55.
4. He could make a stack 13 cans high and would have 9 cans left over. The 13th triangular number is 91 and the 14th is 105.
5. Two consecutive triangular numbers make a square number.
6. Sums of consecutive odd numbers create square numbers: $1 + 3 = 4$, $1 + 3 + 5 = 9$, $1 + 3 + 5 + 7 = 16$

HOME CONNECTION

PA.BLM6a.4 extends the problem explored so far by asking, “Which toy was received in the greatest numbers?”

REVIEWING THE HOME CONNECTION RESULTS

When students return the next day, have them work in pairs to share solutions. Encourage pairs to reach a consensus. Ask for students to volunteer strategies and solutions. Record the strategies and solutions on the board and discuss them. In addition, you might also see the following algorithm:

1 chipmunk \times 10 days = 10 chipmunks	6 ducklings \times 5 days = 30 ducklings
2 blue jays \times 9 days = 18 blue jays	7 rabbits \times 4 days = 28 rabbits
3 puppies \times 8 days = 24 puppies	8 goldfish \times 3 days = 24 goldfish
4 kittens \times 7 days = 28 kittens	9 ladybugs \times 2 days = 18 ladybugs
5 butterflies \times 6 days = 30 butterflies	10 caterpillars \times 1 day = 10 caterpillars

Students may be surprised to learn that there are two solutions to the problem.

ASSESSMENT

ANTICIPATION GUIDE

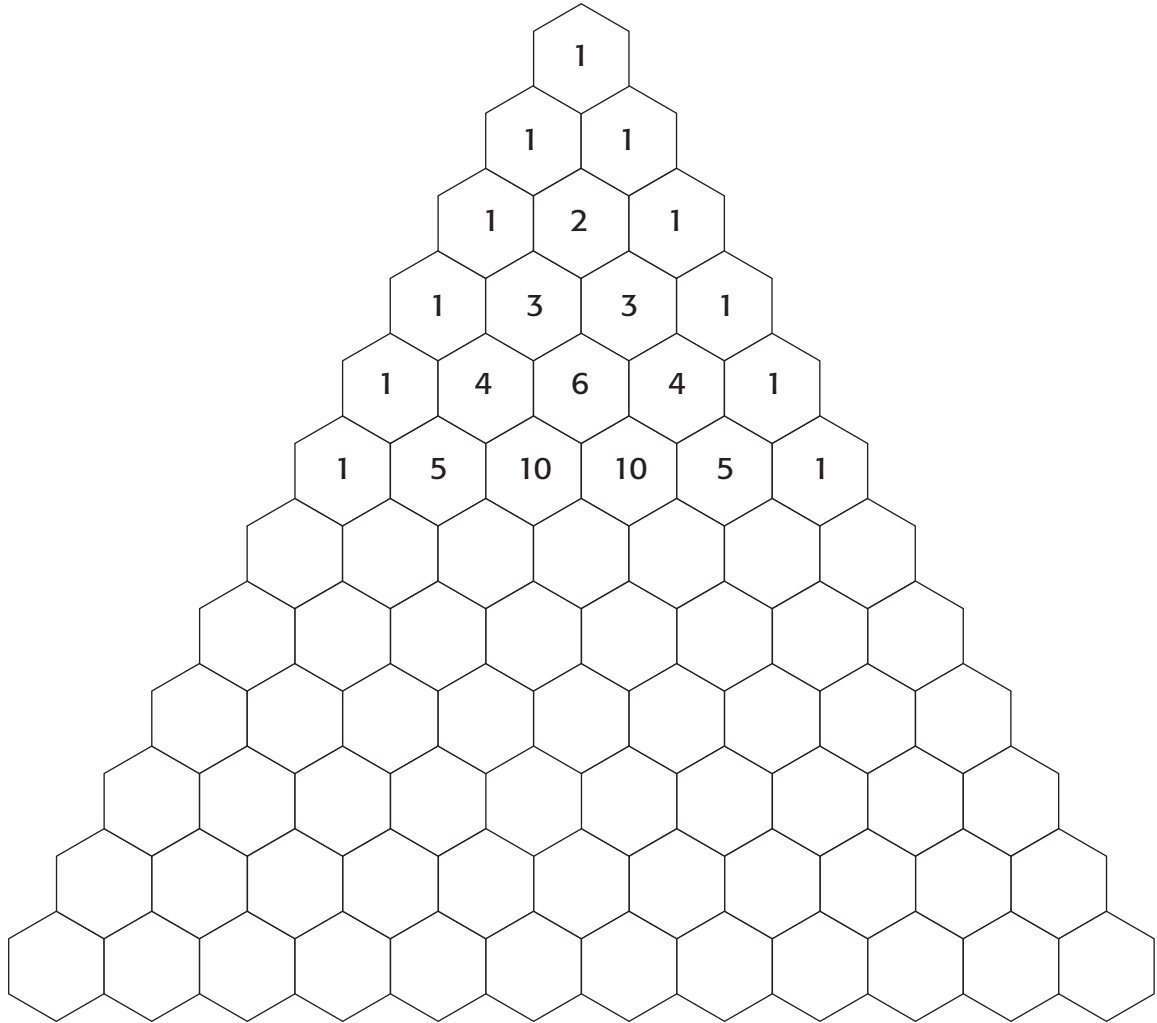
Reconvene the class to discuss the reasons for students’ choices on the “after” section of the Anticipation Guide, and come to a class consensus. Take note of incorrect answers from individual students or large groups. Use this information to determine student understanding and to identify areas of need.

ANTICIPATION GUIDE

Before		Statement	After	
Agree	Disagree		Agree	Disagree
		1. The next number in the sequence 2, 6, 12, 20, ... is 30.		
		2. The next number in the sequence 1, 2, 3, 6, 7, 14, ... is 28.		
		3. Growing patterns always increase by the same number.		
		4. All patterns have rules.		
		5. You can create patterns by using any of the four operations: +, -, ×, ÷.		
		6. The following is a pattern: 3, 7, 10, 22, 24, 30, 29.		

PASCAL'S TRIANGLE

Examine the patterns in Pascal's Triangle. How does Pascal's Triangle grow? How can we use the top two rows to get the third row? How can we use the top three rows to get the fourth row? Fill in the rest of the triangle using the patterns that you notice. Please use pencil.



PROBLEMS CAROUSEL

Cut out the question cards and set them out at carousel centres. Have “table groups” of students move in a clockwise direction around the room, solving each problem. Provide opportunities for students to display and explain their solutions to the problems. The solutions are under Tiered Instruction: Extensions on page 95.

Problem 1: There are 7 people in a group. Each person in the group shakes hands with everyone else exactly once. How many handshakes are there?



Problem 2: The bell known as Big Ben in London, England chimes the hour every hour. At one o'clock it chimes once, at two o'clock it chimes twice, at three o'clock it chimes three times, and so on. How many times does it chime from 12:01 a.m. to 12:01 p.m.?



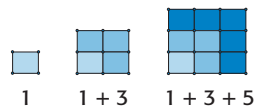
Problem 3: You are making flash cards for a Kindergarten child. You have 10 cards to make. On the first card you write the number 1 and put 1 sticker on the back. On the second card you write the number 2 and put 2 stickers on the back. On the third card you write the number 3 and put 3 stickers on the back. You will continue in this way until you have put 10 stickers on the back of the 10th card. How many stickers will you need to make the cards?

Problem 4: A grocery store wants to stack cans in a triangular display just like the one in the picture. There are 100 cans to put on display. How many rows high can the cans be stacked? Will there be any cans left over?



Problem 5: In an experiment Jamal learned that he could make rectangles when he added two of the same triangular numbers together. He wondered what shape he might get if he added together *consecutive* triangular numbers. Model the problem and discuss the pattern.

Problem 6: Toni noticed a pattern when she added consecutive odd numbers together (starting with 1). What was the pattern?

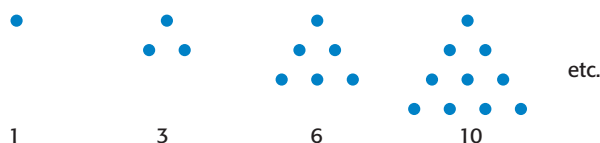


Teacher Note: The cards for each group should be cut out of **PA.BLM6a.3** ahead of time. All but the last card in each set should be paper-clipped together or placed in a sealable plastic baggie. Retain the sixth card for groups that have trouble getting started with the task.

HOME CONNECTION: BEYOND TRIANGULAR NUMBERS

Dear Parent/Guardian:

In math we are exploring many types of patterns. Today we focused on triangular numbers. Triangular numbers are numbers that can form a triangle when represented by dots. This is the triangular pattern:



The home activity builds on an activity we have just completed in class:

At a fun fair, a student wins first prize – a collection of miniature toys. For the next ten days the toys are delivered to the student’s classroom in a most unusual way.

On the first day the winner’s package contains 1 chipmunk; on the second day the package contains 2 blue jays and 1 chipmunk; on the third day the package contains 3 puppies, 2 blue jays, and 1 chipmunk; on the fourth day the package contains 4 kittens, 3 puppies, 2 blue jays, and 1 chipmunk. The same pattern continues until, on the tenth day, the package contains 10 caterpillars, 9 ladybugs, 8 goldfish, 7 rabbits, 6 ducklings, 5 butterflies, 4 kittens, 3 puppies, 2 blue jays, and 1 chipmunk.

Your child needs to answer these two questions:

1. Which toy do you predict will be received in the greatest numbers during those 10 days?

Answer: I predict that the student will have received more _____ than any other miniature toy.

2. Which toy really was delivered in the greatest numbers during those 10 days?

Encourage your child to create a chart or a diagram, or both, to represent his or her solutions. Ask for an explanation of how the answer was found.

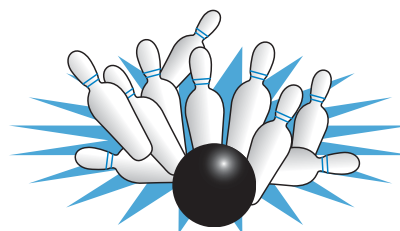
Back in class, students will share these solutions with their classmates.

Grade 6 Learning Activity

Bowling Dilemma

OVERVIEW

In this learning activity students use graphs, tables of values, and equations to compare costs at two different bowling sites, and to determine which site offers the better deal. They also explore the similarities between this problem and one that asks them to compare the top speeds of various animals and to determine which animal would win a race in which the slower animal was given a head start. Students have opportunities to use technology to model the relationships.



Prior to this learning activity students should have had some experience with using variables (symbols or letters), and they should understand the concept of balance in an equation.

BIG IDEA

Variables, expressions, and equations

CURRICULUM EXPECTATIONS

This learning activity addresses the following **specific expectations**.

Students will:

- demonstrate an understanding of different ways in which variables are used (e.g., variable as an unknown quantity; variable as a changing quantity);
- identify, through investigation, the quantities in an equation that vary and those that remain constant (e.g., in the formula for the area of a triangle $A = \frac{1}{2} b \times h$, the number $\frac{1}{2}$ is a constant, whereas b and h can vary and may change the value of A);
- solve problems that use two or three symbols or letters as variables to represent different unknown quantities;
- determine the solution to a simple equation with one variable, through investigation using a variety of tools and strategies (e.g., modelling with concrete materials, using guess and check with and without the aid of a calculator) (**Sample problem:** Use the method of your choice to determine the value of the variable in the equation $2 \times n + 3 = 11$. Is there more than one possible solution? Explain your reasoning.).

These expectations contribute to the development of the following **overall expectation**.

Students will:

- use variables in simple algebraic expressions and equations to describe relationships.

TIME:

1 hour 30 minutes

ABOUT THE LEARNING ACTIVITY

MATERIALS

- graph or chart grid paper
- **PA.BLM6b.1: Bowling Dilemma** (1 per group)
- **PA.BLM6b.2: Home Connection: Constants and Variables** (1 per student)

MATH LANGUAGE

- variable
- constant
- equation
- intersection

INSTRUCTIONAL GROUPING:

table groups (4–6 students) and pairs

ABOUT THE MATH

VARIABLE

A quantity, represented by a letter or symbol, that can have one or more values. For example, in $P = 4s$ (the formula for the perimeter of a square), P and s are variables and 4 is a constant. Variables can be placeholders for missing numbers (as in $\square + \square = 14$ and $2x = 8$), or they can represent quantities that vary (as in $P = 4s$ and $y = 2x + 1$).

CONSTANT

A quantity that stays the same.

ALGEBRAIC EQUATION

A mathematical sentence that contains variables, constants, and an equal sign (e.g., $4x = 12$, $x + y = 15$).

Note: Solving an equation means finding values for the variable(s) that make the equation true.

GETTING STARTED

(Part 1 of the Three-Part Lesson)

WARM UP: THE SPEED OF ANIMALS

Ask students: "Predict which animal runs the fastest."

- Pronghorn antelope
- Garden snail
- House mouse
- Reindeer
- Giant tortoise
- Kangaroo

Teacher note: For the top speeds of various animals, do an Internet search using the key words "animal speeds". Here are the (approximate) top speeds of the animals in this problem:

- pronghorn antelope: 28 m/s
- kangaroo: 14 m/s
- reindeer: 4 m/s
- house mouse: 2 m/s
- giant tortoise: 3 m/min
- garden snail: 0.5 m/min

Allow students time to share their prediction with their elbow partner. Reveal the relationships between the running speeds of the animals and ask students to compare it to their prediction. “A pronghorn antelope can run twice as fast as a kangaroo. A reindeer can run 4 times as fast as a house mouse. A giant tortoise can move 6 times as fast as a garden snail.” Ask: “Suppose that an antelope and kangaroo had a race, and the kangaroo had a head start of 100 m. Who would win the race?”

Ask students to work in pairs on the problem, then discuss solution ideas as a whole class. Draw attention to the various methods that students use to analyse the problem, such as drawing a diagram, making a T-chart, or acting out the situation. Students will notice that the length of the race has not been specified. Ask them to consider different race lengths to see if length makes a difference.

WORKING ON IT

(Part 2 of the Three-Part Lesson)

BOWLING DILEMMA

Distribute **PA.BLM6b.1** to students and do a shared reading of the scenario with the class:

You want to invite your friends to a bowling competition. You haven’t yet decided how many games will be played in the competition, but you are confident that you will play at least 2 games and at most 5 games. There are two bowling sites quite near by, and you are trying to decide which site offers the better deal. At site 1 each person would pay a rate of \$3.50 per game plus a one-time \$3.00 fee for shoe rental. At site 2 each person would pay a rate of \$4.50 per game, but there is no shoe rental fee. Which site is the better value?

CREATING ALGEBRAIC EQUATIONS TO SOLVE THE PROBLEM

Using a think-aloud method, pose the following statements to the class before the students begin working on the problem:

“I think”:

- I need an equation for the cost of playing bowling at site 2.
- The cost itself is one of the variables. I’ll call this variable C . My equation so far is: $C = \dots$

“I wonder”:

- What other variables are there? What letters could I use to represent them?
- What constants are there?
- How can I write the equation?
- How I will test the equation?

Have students work in pairs to solve the problem. After students have worked for a while and have developed some methods of representing the different sites, create a chart and record the students’ ideas on how to represent each site.

Site 1	Site 2
$C = 3.50 \times G + 3$	$C = 4.50 \times G$
<ul style="list-style-type: none"> • C represents the cost • \$3.50 is the cost per game • G represents the number of games that will be played • \$3.00 is the equipment rental fee 	<ul style="list-style-type: none"> • C represents the cost • \$4.50 is the cost per game • G represents the number of games that will be played

Scaffolding suggestion:
Provide the headings for each column.

Once most pairs of students have developed their second equation, discuss the equation as a whole class, retracing the steps modelled above.

CREATING A TABLE OF VALUES

Explain to students that they will now be calculating the costs at each site using the equations they have just developed. With student assistance, model how to use the equations to calculate the cost of one game at each site, and enter the data in a table of values such as the one at right. Complete a row as a whole class, and then send students back to their small groups or pairs to continue their work. Ask them to include a table of values in their solutions.

Number of Games	Site 1 (\$)	Site 2 (\$)
1	$3.50(1) + 3$ $= 6.50$	$4.50(1)$ $= 4.50$
2		
3		
4		
5		

LOOKING BACK

Review students' solutions and select two or three to show how students used the table of values. Explain that another way to represent this information is by using a graph. See **PA.BLM6b.1**.

Instruct students to plot both costs on the same graph so that they can see where the costs intersect. Students could use graph or chart grid paper or spreadsheet software to create line graphs based on the data in the table. Ask students to compare their table of values and their graph to see if the table and graph support their answer to the original problem. Ask them to explain their thinking to another group.

Number of Games	Site 1 (\$)	Site 2 (\$)
1	6.50	4.50
2	10.00	9.00
3	13.50	13.50
4	17.00	18.00
5	20.50	22.50

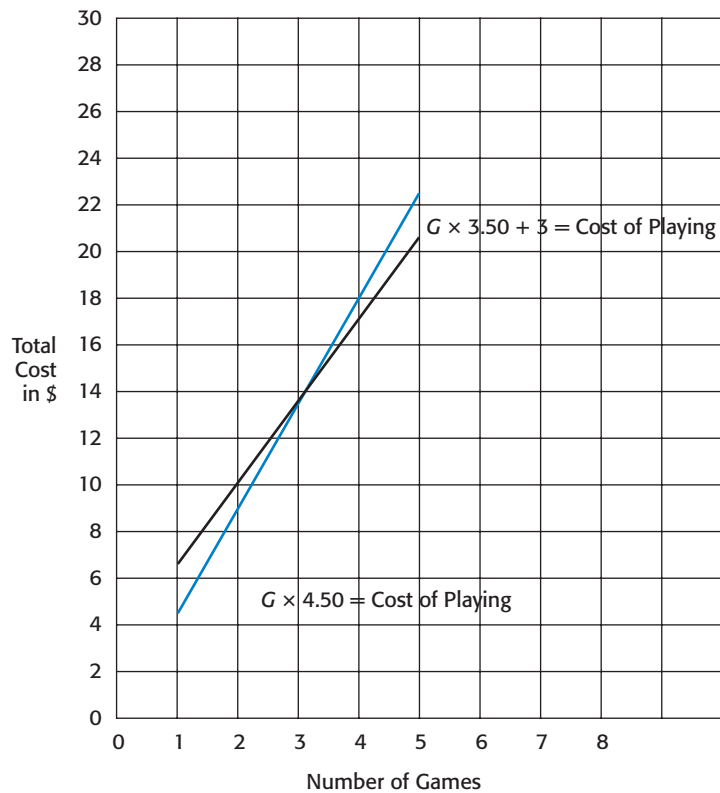
(The table and the graph shown here and on the previous page compare the costs of playing up to 5 games. Site 2 offers the better deal until it intersects with site 1 at the 3rd game. After this, site 1 becomes the better choice. The cost is the same for both sites at the point where the two graph lines intersect.)

REFLECTING AND CONNECTING

(Part 3 of the Three-Part Lesson)

Facilitate a class discussion of the bowling problem. Ask:

- Is there just one solution?
- Which site would you pick? Why?
- What does the intersection point on the graph represent?



TIERED INSTRUCTION

Supports and extensions can be beneficial for all students. For any given activity, there will always be some students who require more or less support, or for whom extensions will increase interest and deepen understanding.

SUPPORTS FOR STUDENT LEARNING

- **Bowling dilemma.** Give students separate charts to generate ordered pairs.
- **Equations.** Use different colours for the variable(s) and the constant.

EXTENSIONS

Revisiting The Speed Of Animals Problem. Present to students the (approximate) top speeds of the animals used in the Speed of Animals problem:

- pronghorn antelope: 28 m/s
- kangaroo: 14 m/s
- reindeer: 4 m/s
- house mouse: 2 m/s
- giant tortoise: 3 m/min
- garden snail: 0.5 m/min

Pose the following problem:

“Suppose that an antelope and a kangaroo had a race, and the kangaroo had a head start of 100 m. Who would win the race?”

Ask them to work in pairs to do the following:

- Create algebraic equations to model the speed of each animal.
- Create a chart showing the distance travelled by each animal over a period of 10 seconds.
- Identify the constants and variables in the expressions.

Contrast the equations and the graphs for the two problems. What similarities do you notice? What differences are there?

Facilitate a class discussion of the ideas that emerge.

Bowling dilemma extension. Challenge students to create an equation for a third site, whose costs will fall between the costs of the other two.

HOME CONNECTION

See **PA.BLM6b.2: Home Connection: Constants and Variables.**

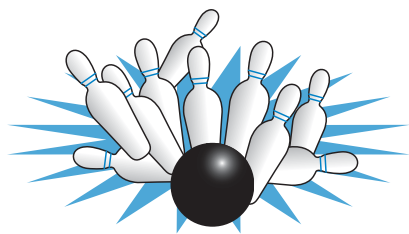
Assessment opportunities might include:

- paragraphs in a reflective journal, completed at school, which explain the Home Connection poster activity;
- informal discussions with students during the activity.

BOWLING DILEMMA

You want to invite your friends to a bowling competition. You haven't yet decided how many games will be played in the competition, but you are confident that you will play at least 2 games and at most 5 games. There are two bowling sites quite near by, and you have to decide which site offers the better deal. At site 1 each person would pay a rate of \$3.50 per game plus a one-time \$3.00 fee for shoe rental. At site 2 each person would pay a rate of \$4.50 per game but there is no shoe rental fee. Which site is the better value?

- Create algebraic equations to model the cost at each site.
- Create a chart showing the cost of the first 5 games for both sites.
- Identify the constants and variables in the expressions.
- Plot both sets of data on a line graph. Use a different colour for each line. Where do the lines intersect (cross)? Explain what the graph shows.
- Which site is the better value? What assumptions have you made?
- If you played 20 games at site 1, how much money would you have to pay? Explain your reasoning.

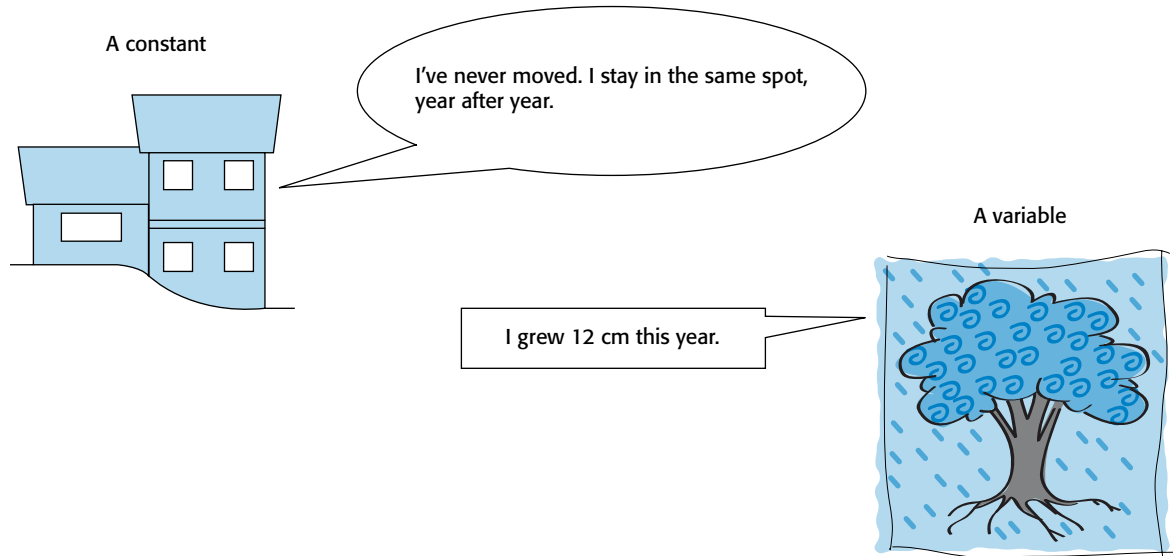


HOME CONNECTION: CONSTANTS AND VARIABLES

Dear Parent/Guardian:

In class we have been studying constants and variables in equations. We have explored relationships between constants and variables and have shown them in the form of charts, graphs, and algebraic equations.

Please ask your child to create a poster using numbers, pictures, and words to illustrate the concept of constants and variables. For example:



APPENDIX: GUIDELINES FOR ASSESSMENT

There are three types of assessment: assessment *for* learning, assessment *as* learning, and assessment *of* learning.

Assessment *for* learning involves teachers observing the knowledge, skills, experience, and interests their students demonstrate, and using those observations to tailor instruction to meet identified student needs and to provide detailed feedback to students to help them improve their learning.

Assessment *as* learning is a process of developing and supporting students' metacognitive skills. Students develop these skills as they monitor their own learning, adapt their thinking, and let the ideas of others (peers and teachers) influence their learning. Assessment *as* learning helps students achieve deeper understanding.

Assessment *of* learning is summative. It includes cumulative observations of learning and involves the use of the achievement chart to make judgements about how the student has done with respect to the standards. Assessment *of* learning confirms what students know and are able to do, and involves reporting on whether and how well they have achieved the curriculum expectations.

Teachers use assessment data, gathered throughout the instruction–assessment–instruction cycle, to monitor students' progress, inform teaching, and provide feedback to improve student learning. Effective teachers view instruction and assessment as integrated and simultaneous processes. Successful assessment strategies – those that help to improve student learning – are thought out and defined ahead of time in an **assessment plan**.

An **assessment profile**, developed by the teacher for each student, can be an effective way of organizing assessment data to track student progress. Students can also maintain their own **portfolios**, in which they collect samples of their work that show growth over time.

Creating an Assessment Plan

To ensure fair and consistent assessment throughout the learning process, teachers should work collaboratively with colleagues to create assessment and instructional plans. Ideally, such planning should start with learning goals and work backwards to identify the assessment and instructional strategies that will help students achieve those goals.

Guiding Questions	Planning Activities
"What do I want students to learn?"	Teachers begin planning by identifying the overall and specific expectations from the Ontario curriculum that will be the focus of learning in a given period. The expectations may need to be broken down into specific, incremental learning goals. These goals need to be shared with students, before and during instruction, in clear, age-appropriate language.
"How will I know they have learned it?"	Teachers determine how students' learning will be assessed and evaluated. Both the methods of assessment and the criteria for judging the level of performance need to be shared with students.
"How will I structure the learning?"	Teachers identify scaffolded instructional strategies that will help students achieve the learning goals and that integrate instruction with ongoing assessment and feedback.

An assessment plan should include:

- clear learning goals and criteria for success;
- ideas for incorporating both assessment *for* learning and assessment *as* learning into each series of lessons, before, during, and after teaching and learning;
- a variety of assessment strategies and tools linked carefully to each instructional activity;
- information about how the assessment profiles will be organized;
- information about how the students' assessment portfolios will be maintained.

Feedback

When conducting assessment *for* learning, teachers continuously provide timely, descriptive, and specific feedback to students to help them improve their learning. At the outset of instruction, the teacher shares and clarifies the learning goals and assessment criteria with the students. Effective feedback focuses the student on his or her progress towards the learning goals. When providing effective feedback, teachers indicate:

- what good work looks like and what the student is doing well;
- what the student needs to do to improve the work;
- what specific strategies the student can use to make those improvements.

Feedback is provided during the learning process in a variety of ways – for example, through written comments, oral feedback, and modelling. A record of such feedback can be maintained in an assessment profile.

Assessment Profile

An assessment profile is a collection of key assessment evidence, gathered by the teacher over time, about a student's progress and levels of achievement. The information contained in the profile helps the teacher plan instruction to meet the student's specific needs. An extensive collection of student work and assessment information helps the teacher document the student's progress and evaluate and report on his or her achievement at a specific point in time.

The assessment profile also informs the teacher's conversations with students and parents about the students' progress. Maintaining an assessment profile facilitates a planned, systematic approach to the management of assessment information.

Assessment profiles may include:

- assessments conducted after teaching, and significant assessments made during teaching;
- samples of student work done in the classroom;
- samples of student work that demonstrates the achievement of expectations;
- teacher observation and assessment notes, conference notes;
- EQAO results;
- results from board-level assessments;
- interest inventories;
- notes on instructional strategies that worked well for the student.

Student Assessment Portfolio

A portfolio is a collection of work selected by the student that represents his or her improvement in learning. It is maintained by the student, with the teacher's support. Assembling the portfolio enables students to engage actively in assessment *as* learning, as they reflect on their progress. At times, the teacher may guide students in the selection of samples that show how well they have accomplished a task, that illustrate their improvement over a period of time, or that provide a rationale for the teacher's assessment decisions. Selections are made on the basis of previously agreed assessment criteria.

Student assessment portfolios can also be useful during student/teacher and parent/teacher conferences. A portfolio may contain:

- work samples that the student feels reflect growth;
- the student's personal reflections;
- self-assessment checklists;
- information from peer assessments;
- tracking sheets of completed tasks.

Students should not be required to assign marks, either to their own work or to the work of their peers. Marking is part of the evaluation of student work (i.e., judging the quality of the work and assigning a mark) and is the responsibility of the teacher.

Assessment Before, During, and After Learning

Teachers assess students' achievement at all stages of the instructional and assessment cycle.

Assessment *before* new instruction identifies students' prior knowledge, skills, strengths, and needs and helps teachers plan instruction.

Effective assessment *during* new instruction determines how well students are progressing and helps teachers plan required additional instruction. The teacher uses a variety of assessment strategies, such as focused observations, student performance tasks, and student self- and peer assessment, all based on shared learning goals and assessment criteria. As noted earlier, the teacher provides students with feedback on an ongoing basis during learning to help them improve.

During an instructional period, the teacher often spends part of the time working with small groups to provide additional support, as needed. The rest of the time can be used to monitor and assess students' work as they practise the strategies being learned. The teacher's notes from his or her observations of students as they practise the new learning can be used to provide timely feedback, to develop students' assessment profiles, and to plan future lessons. Students can also take the opportunity during this time to get feedback from other students.

Assessment *after* new learning has a summative purpose. As assessment *of* learning, it involves collecting evidence on which to base the evaluation of student achievement, develop teaching practice, and report progress to parents and students. After new learning, teachers assess students' understanding, observing whether and how the students incorporate feedback into their performance of an existing task or how they complete a new task related to the same learning goals. The assessment information gathered at this point, based on the identified curriculum expectations and the criteria and descriptors in the achievement chart, contributes to the evaluation that will be shared with students and their parents during conferences and by means of the grade assigned and the comments provided on the report card.

Glossary

array A rectangular organization of objects. For example, the 9 dots that represent the number 9 can be shown in a 3×3 array.



bar graph *See under* graph

calculus The algebraic study of rates of change.

Cartesian coordinate grid or **Cartesian plane** *See* coordinate plane

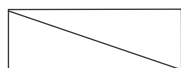
constant A part of an expression that does not change. For example, in the expression $m + 1$, the number 1 is a constant, while m is a variable.

coordinate graph *See under* graph

coordinate plane A plane that contains an x -axis (horizontal) and a y -axis (vertical), which are used to describe the location of a point. Also called Cartesian coordinate grid or Cartesian plane.

coordinates An ordered pair used to describe location on a grid or plane. For example, the coordinates (3, 5) describe a location found by moving 3 units to the right and 5 units up from the origin (0, 0). *See also* ordered pair

diagonal A line segment that joins two non-adjacent vertices of a polygon.



double bar graph *See under* graph

double line graph *See under* graph

equation A mathematical statement that has equivalent expressions on either side of an equal sign.

exponent *See under* exponential form

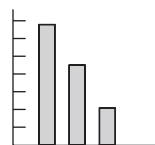
exponential form A representation of a product in which a number called the base is multiplied by itself. The exponent is the number of times the base appears in the product. For example, 5^4 is in exponential form, where 5 is the base and 4 is the exponent; 5^4 means $5 \times 5 \times 5 \times 5$.

expression A numeric or algebraic representation of a quantity. An expression may include numbers, variables, and operations; for example, $3 + 7$, $2x - 1$.

general term of a sequence An algebraic expression that represents the rule for determining the terms of a sequence. For example, $2 \times n + 1$ is the general term of the sequence 3, 5, 7, 9,

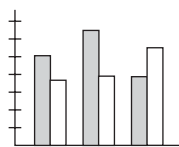
graph A visual representation of data. Some types of graphs are:

– **bar graph** A graph consisting of horizontal or vertical bars that represent the frequency of an event or outcome. There are gaps between the bars to reflect the categorical or discrete nature of the data.



– **coordinate graph** A graph that has data points represented as ordered pairs on a grid; for example, (4, 3). *See also* ordered pair

– **double bar graph** A graph that combines two bar graphs to compare two aspects of the data in related contexts; for example, comparing the populations of males and females in a school in different years. Also called comparative bar graph.



– **double line graph** A graph that combines two line graphs to compare two aspects of the data in related contexts; for example, comparing the distance travelled by two cars moving at different speeds.

– **line graph** A graph formed by (a) straight line(s).

grid A network of regularly spaced lines that cross one another at right angles to form squares or rectangles.

growth pattern A pattern whose terms represent growing quantities.

identity A mathematical statement that is always true. For example, $3 + 5 = 8$ and $x + y = y + x$ are identities.

infinite Having no limit; larger than any (finite) number.

intersection point The point, or coordinates, where two lines meet on a coordinate grid.

line graph See under graph

linear pattern A numeric pattern in which numbers grow at a constant rate. For example, 3, 5, 7, 9, 11,

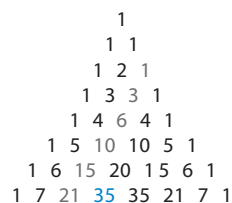
missing numbers The numbers needed to solve an equation, such as $3 + \square = 8$ and $5 + n = 12$.

natural numbers The numbers 1, 2, 3, 4,

***n*th term of a sequence** See general term of a sequence

ordered pair Two numbers, in order, that are used to describe the location of a point on a plane, relative to a point of origin (0, 0); for example, (2, 6). On a coordinate plane, the first number is the horizontal coordinate of a point, and the second is the vertical coordinate of the point. See also coordinates

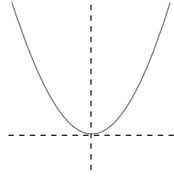
Pascal's Triangle Pascal's Triangle is named after Blaise Pascal, the French mathematician who studied it. The pattern was known in China long before Pascal's time. Notice how Pascal's Triangle grows:



- the numbers on the outer diagonals have a value of 1;
- each of the other numbers is the sum of the two numbers directly above it (for example, the 6 in the 5th row is the sum of the two 3's above it).

Pascal's Triangle has many applications in mathematics and contains within itself many interesting patterns, such as triangular numbers.

quadratic function A function of the form $f(x) = ax^2 + bx + c$. The shape of its graph is a parabola (like the path of a thrown object). The graph of the vertical distance travelled by a falling object is (approximately) parabolic.



square numbers Square numbers are formed by multiplying a number by itself: 1, 4, 9, 16, 25, 36, 49,

T-chart A table of values with two columns (input and output or term number and term).

tally chart A chart that uses tally marks to count data and record frequencies.

temporal Having to do with time.

term Each of the quantities constituting a ratio, a sum or difference, or an algebraic expression. For example, in the ratio 3:5, 3 and 5 are both terms; in the algebraic expression $3x + 2y$, $3x$ and $2y$ are both terms.

triangular numbers The numbers 1, 3 (= $1 + 2$), 6 (= $1 + 2 + 3$), 10 (= $1 + 2 + 3 + 4$), ...; they appear in the 3rd diagonal of Pascal's Triangle.

variable A letter or symbol used to represent an unknown quantity, a changing value, or an unspecified number (e.g., $a \times b = b \times a$).

x-axis The horizontal number line on a coordinate plane.

y-axis The vertical number line on a coordinate plane.



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