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What Works? Research into Practice

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Research Monograph # 2

How can teachers help students acquire a deep understanding of mathematics?

Learning Mathematics vs Following "Rules": The Value of Student-Generated Methods

Research Tells Us

- Most (but not all) children will, with extensive practice, eventually learn to use traditional algorithms with some competence. This skill often comes at too great a price: students learn to ignore their own reasoning in favour of following rules.
- Students who are encouraged to experiment with their own solutions to mathematics problems develop a significantly deeper understanding of mathematics.
- These students are also less prone to make errors than are their counterparts in traditional mathematics classrooms.
- Such methods are helpful for all students – including those struggling with their mathematics.

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There has been a significant shift in the instruction of mathematics over the past two decades. This shift has occurred in response to growing evidence that students were learning how to apply mathematics rules without a real understanding of the mathematics.[1] A particularly disconcerting observation was that student difficulties often stemmed from our longstanding traditional methods of mathematics instruction.[2]

Typical Errors that Students Make

Paul Cobb's[3] study of children learning double-digit addition in Grade 2 characterizes these findings. He conducted mathematical interviews with Grades 1 and 2 children asking, "Do you have a way to figure out how much is 16 + 9?" All of them were able to find the answer of 25 using a range of methods such as counting on. Later, students were given the same problem embedded in a typical school text with a vertical format. This time many children attempted to use the standard school algorithm with carrying. Many made errors – ones that primary teachers would recognize – such as 15 or 115. When Cobb discussed the answer of 15 with one child and asked her whether both her original 25 could be right as well as her later answer of 15, she said: "If you were counting cookies [at home] 25 would be right … in school, it (points to her answer of 15 on the worksheet) is always right".

Indeed the research literature is replete with examples of "buggy" algorithms. These are typical errors that students make as they try to follow rules such as "carrying" or "borrowing" which they do not really understand. Nagel and Swingen[4] examined 52 elementary students' use of traditional algorithms to solve double-digit addition and subtraction problems requiring regrouping.

What follows are some of the typical errors.

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Figure 1. "Buggy" Errors in Children's Double-Digit Addition and Subtraction Adapted from Nagel & Swingen, 1998, p. 167

Computation	Child's Solution
28 +29 471	Alena worked from the left side of the numbers. She added 2 plus 2 and placed 4 under these numbers. She then added 8 plus 9 and wrote 71, reversing the 17 under this column, coming to an answer of 471.
76 -29 53	Andy said, "I took away 2 from the 7, then got 5. Then I took 6 from the 9 and got 3." His answer was 53, which resulted from looking at the ones column and finding the smaller number to take away, regardless of number placement.
76 511 -29 -29 32	Kelly thought that she could not subtract 9 from 6 so she changed the 7 to a 5, "borrowed" 11 (two 1s) from the 10s column and placed it in the units' column, and then subtracted.

Children have difficulty making sense of our traditional North American algorithms for good reason. These algorithms were developed over time to maximize efficiency and accuracy before the time of calculators.[5] They were not meant to maintain sense-making for the learner; instead, they embody many shortcuts based upon extensive mathematics – mathematics often beyond the capacity of the average Grade 2 student. Therefore, there has been an important shift to improve understanding by beginning instruction using children's initial understandings.

Children's Solution Strategies

More than two decades ago Tom Carpenter and his research team[6] began asking children to solve problems without the benefit of direct instruction of methods. They found that children would generate a variety of solution strategies when given, for example, a primary division problem such as this: *Maria's mom baked 42 cupcakes. She is placing them in 7 tins. If she puts the same number in each how many should she place in a tin?*

- At first, most children will model the problem directly by counting out 42 items or drawing 42 cupcakes, then drawing 7 tins and "doling" out the cupcakes into the tins one at a time until they run out. Alternatively, they may count out or draw groups of 7 cupcakes, adding more groups of 7 as needed until they get to 42, and then recounting the groups to determine the number in each tin. At this stage they are concretely modelling the action of the problem.

- Children will learn to replace these direct or concrete modelling procedures with counting strategies, often skip-counting and keeping track on their fingers how many times they counted. Or they may instead add in some fashion, perhaps doubling (7 + 7 = 14), then doubling again (14 + 14 = 28), and then adding (14 + 28 = 42). Children often find doubling easier than other forms of addition.

- Later, they may use derived facts to solve the problem. That is, as they construct some multiplication facts they know, such as fives (which are easier), they use this to derive ? x 7 = 42. If you know that 5 x 7 = 35, then one more 7 will work. Eventually, most children will solve this as a division fact (42 ÷ 7 = 6).

There are many other long-term research projects with similar findings, such as Karen Fuson's Supporting Ten-Structured Thinking Projects[7] and Constance Kamii's ongoing work in Children Reinventing Arithmetic.[8,9] Students in these classrooms have a significantly deeper understanding and enjoyment of the mathematics than their counterparts in traditional instruction classrooms.

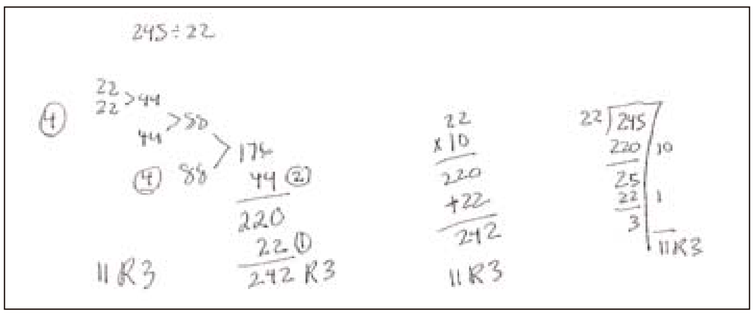
These ideas are now being extended into the junior grades.[10,11] Students who develop a strong foundation in solving problems using their own methods at the primary level can use this knowledge to tackle more challenging problems in the junior grades. While we have somewhat less research on the effectiveness of encouraging student-generated or alternative algorithms at the junior level than the primary, there is mounting evidence that this approach continues to promote deeper understanding and fewer misconceptions or errors than is the case with direct instruction of standard algorithms.[12,13] Moreover there is evidence that these methods are more accessible for all – including students struggling with their mathematics.[14]

What Teachers Can Do to Support Mathematics Learning

It is important to note that this progression of student strategies from early concrete modelling through to efficient, alternative or standard algorithms is neither linear nor developmental. Instead the progression is experiential – the result of classroom experiences in which teachers effectively support children in solving problems using their own methods. How do teachers do this?

At the junior level, for example, it would mean introducing division with an accessible problem, set in a familiar context, rather than as a series of steps to be learned.

Figure 2. Student-Generated Methods and Alternative Algorithm for 245 ÷ 22



- **Pose a problem** such as: *I have a large bag of 245 M&Ms. If we divide this evenly among the class (22 students), how many would each of you get?*

- **Pair students with a partner who is at the same mathematical level** in order to encourage full participation of both students.

- **Allow students to try to solve the problem with the method that makes sense to them**. Students may add up, multiply, subtract, or divide to solve this problem (see the first two examples in Figure 2 for typical solutions). Students will likely require a full period to solve their problem and get ready to share their ideas during the math discussion or "congress" at a later time. Choose a few pairs to share their thinking with the class.

- **Introduce a math congress to focus student thinking on one or two strategies or perhaps "Big Ideas."** For example, when students solve a division problem (such as the M&M problem) using different strategies it is an opportunity to ask: Why is it we can multiply or divide to find the same answer? The teacher can make use of students' varied solutions to explore the Big Idea that multiplication is the inverse of division.[15] In addition to helping students learn to calculate with greater understanding and capacity, these methods also allow teachers to capitalize on children's thinking in order to deepen their knowledge of mathematics – a capitalization not available when students are restricted to the traditional method or calculator.

Implications for Educational Practice

For More Discussion ...

- Ontario Ministry of Education (2003). *Early math strategy: The report of the Expert Panel on Early Math in Ontario*. Queen's Printer.

- Ontario Ministry of Education (2004). *Teaching and learning mathematics: The report of the Expert Panel on Mathematics in Grades 4–6 in Ontario*. Queen's Printer. See pp. 7–21. http://www.edu.gov.on.ca/eng/document/reports/numeracy/panel/numeracy.pdf

- van de Walle, J. (2007). *Elementary and middle school mathematics: Teaching developmentally.* Sixth Ed. New York: Longman. See pp. 21–224.

For an Ontario Classroom in Action ...

- Miller, L. (2004). Brookmede Public School and École catholique St. Antoine share tips for inspiring young math minds. *Professionally Speaking*. http://www.oct.ca/publications/professionally_speaking/june_2004/math.asp

- **Give students many opportunities to solve different division problems** so that they will slowly progress towards multiplying up or subtracting greater "chunks" or copies of the divisor. At this time you could introduce the "Dutch"[16] or "accessible division"[12] or "alternative division"[11] algorithm as a way to structure their thinking (see the final solution in Figure 2). If students are already taking away or multiplying and adding up larger multiples of the divisor, this is a structure that will make sense to them and be easily adopted. Finally, students will likely also bring in traditional algorithms that can be explored to examine the mathematics and learn why and how they work.

It must be stated that this is a highly demanding approach requiring mathematical and instructional knowledge, perseverance and patience. Many teachers express doubts about their ability to teach, and their students' ability to learn, mathematics in this fashion.[17] They often find the first few weeks particularly challenging as children, and sometimes their parents, expect direct instruction of algorithms – mathematics as they knew it.[18] The eventual rewards of these instructional changes, however, are classrooms where more children genuinely understand and enjoy mathematics than has been the norm.

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Professional Learning

The Literacy and Numeracy Secretariat has developed a range of resources to help classroom teachers enhance their mathematical knowledge and understanding:

- **Numeracy Professional Learning Series** on teaching addition and subtraction, multiplication and division, fractions and per cents, and learning through problem solving
  www.curriculum.org/LNS/coaching
- **Mathematical Knowledge for Teaching**
  Webcast featuring Deborah Loewenberg Ball
  www.curriculum.org/secretariat/november2.html
- **Making Mathematics Accessible to All Students**
  Webcast featuring Mary Lou Kestell, Kathryn Kubota-Zarivnij, and Marian Small
  Available as of March 30, 2007
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